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A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. Contributors are requested to bear in mind that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with earlier results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

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A FORMULA FOR THE SPURIOUS CORRELATION BETWEEN TWO FUNCTIONS OF MUTUALLY UNCORRELATED VARIABLES.

BY N. K. ADYANTHAYA, B.A. HONS. B. SC. (Lond.),
Lecturer in Statistics, University of Madras.

In Vol. LX of the *Proceedings of the Royal Society of London* Professor Karl Pearson gave a formula for the spurious correlation between two indices $\frac{x}{y}$, and $\frac{z}{y}$. The object of this paper is to give a general formula for the spurious correlation between two functions and to make a few deductions therefrom.

1. *Lemma.* If $u = f(x_1\, x_2)$, and $v = \phi\,(x_3, x_4)$ be two functions, their product moment is approximately equal to

$$\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_3} r_{13}\,\sigma_1\sigma_3 + \frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_4} r_{14}\,\sigma_1\sigma_4 + \frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_3} r_{23}\,\sigma_2\sigma_3 + \frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_4} r_{24}\,\sigma_2\sigma_4$$

where r_{13}, r_{23}, etc., are the co-efficients of correlation between x_1 and, x_3, x_2 and x_3, etc., and σ_1, σ_2, etc., are the standard deviations of x_1, x_2 etc.

Now $$u = f(x_1 . x_2) = f(\bar{x}_1 + \delta x_1, \bar{x}_2 + \delta x_2)$$

where $\bar{x}_1$ and $\bar{x}_2$ are the means of x_1 and x_2 respectively and δx_1, δx_2 are the deviations from these means

$$= f(\bar{x}_1, \bar{x}_2) + \delta x_1 \frac{\partial u}{\partial \bar{x}_1} + \delta x_2 \frac{\partial u}{\partial \bar{x}_2} +$$

$$\tfrac{1}{2}\left[(\delta x_1)^2 \frac{\partial^2 u}{\partial \bar{x}_1^{\,2}} + 2\delta x_1\, \delta x_2 \frac{\partial^2 u}{\partial \bar{x}_1\, \partial \bar{x}_2} + (\delta x^2)^2 \frac{\partial^2 u}{\partial \bar{x}_2^{\,2}}\right] + \ldots \quad (1)$$

neglecting terms of the third and higher orders

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[In the derivatives, $\bar{x}_1$ $\bar{x}_2$ are supposed to have been substituted for x_1 x_2 in u].

Let $\{u\}$ denote the mean value of the function u.

Taking mean values of both sides of equation (1),

$$\{u\} = f(\bar{x}_1, \bar{x}_2) + \tfrac{1}{2}\left[\frac{\partial^2 u}{\partial \bar{x}_1^2}\sigma_1^2 + 2\frac{\partial^2 u}{\partial \bar{x}_1 \partial \bar{x}_2} r_{12}\sigma_1\sigma_2 + \frac{\partial^2 u}{\partial \bar{x}_2^2}\sigma_2^2\right] \quad \ldots \quad (2)$$

up to terms of the 2nd order.

Similarly,

$$\{v\} = f(\bar{x}_3, \bar{x}_4) + \tfrac{1}{2}\left[\frac{\partial^2 v}{\partial \bar{x}_3^2}\sigma_3^2 + \frac{2\partial^2 v}{\partial \bar{x}_3 \partial \bar{x}_4} r_{34}\sigma_3\sigma_4 + \frac{\partial^2 v}{\partial \bar{x}_4^2}\sigma_4^2\right] \quad \ldots \quad (3)$$

The product moment of the functions u, v is $\{uv\} - \{u\}\{v\}$. This, after some simplification, can be easily seen to be equal to

$$\frac{\partial u}{\partial \bar{x}_1}\frac{\partial v}{\partial \bar{x}_3} r_{13}\sigma_1\sigma_3 + \frac{\partial u}{\partial \bar{x}_1}\frac{\partial v}{\partial \bar{x}_4} r_{14}\sigma_1\sigma_4 + \frac{\partial u}{\partial \bar{x}_2}\frac{\partial v}{\partial \bar{x}_3} r_{23}\sigma_2\sigma_3 + \frac{\partial u}{\partial \bar{x}_2}\frac{\partial v}{\partial \bar{x}_4} r_{21}\sigma_2\sigma_4 \quad \ldots \quad (4)$$

up to terms of the 2nd order.

The result can be easily extended to functions of n variables.

2. Now, make the functions f and ϕ identical and $x_1 \equiv x_3$ and $x_2 \equiv x_4$.

Then, $\sigma_u^2 = \{u^2\} - \{u\}^2$

$$= \left(\frac{\partial u}{\partial \bar{x}_1}\right)^2 \sigma_1^2 + \left(\frac{\partial u}{\partial \bar{x}_2}\right)^2 \sigma_2^2 + \frac{2\partial u}{\partial \bar{x}_1}\frac{\partial u}{\partial \bar{x}_2} r_{12}\sigma_1\sigma_2 \quad \ldots \quad (5)$$

Similarly,

$$\sigma^2_v = \left(\frac{\partial v}{\partial \bar{x}_3}\right)^2 \sigma_3^2 + \left(\frac{\partial v}{\partial \bar{x}_4}\right)^2 \sigma_4^2 + 2\frac{\partial v}{\partial \bar{x}_3}\frac{\partial v}{\partial \bar{x}_4} r_{34}\sigma_3\sigma_4 \quad \ldots \quad (6)$$

up to terms of the 2nd order.

These two formulæ give the variances of the functions u, v respectively.

3. The next step is to find the correlation between the two functions. This is equal to

$$\frac{\{uv\} - \{u\}\{v\}}{\sigma_u \sigma_v}$$

$$= \frac{\frac{\partial u}{\partial \bar{x}_1}\frac{\partial v}{\partial \bar{x}_3} r_{13}\sigma_1\sigma_3 + \frac{\partial u}{\partial \bar{x}_1}\frac{\partial v}{\partial \bar{x}_4} r_{14}\sigma_1\sigma_4 + \frac{\partial u}{\partial \bar{x}_2}\frac{\partial v}{\partial \bar{x}_3} r_{23}\sigma_2\sigma_3 + \frac{\partial u}{\partial \bar{x}_2}\frac{\partial v}{\partial \bar{x}_4} r_{24}\sigma_2\sigma_4}{\sqrt{\left(\frac{\partial u}{\partial \bar{x}_1}\right)^2\sigma_1^2 + \left(\frac{\partial u}{\partial \bar{x}_2}\right)^2\sigma_2^2 + 2\frac{\partial u}{\partial \bar{x}_1}\frac{\partial u}{\partial \bar{x}_2} r_{12}\sigma_1\sigma_2} \times \sqrt{\left(\frac{\partial v}{\partial \bar{x}_3}\right)^2\sigma_3^2 + \left(\frac{\partial v}{\partial \bar{x}_4}\right)^2\sigma_4^2 + 2\frac{\partial v}{\partial \bar{x}_3}\frac{\partial v}{\partial \bar{x}_4} r_{34}\sigma_3\sigma_4}} \quad \ldots \quad (7)$$

If the functions f, ϕ are identical, and if further $r_2 \equiv x_4$, the correlation between the two functions $u = f(x_1, x_2)$ and $v = f(x_3, x_2)$ is given by the formula.

$$\frac{\frac{\partial u}{\partial \bar{x}_1}\frac{\partial v}{\partial \bar{x}_3} r_{13}\sigma_1\sigma_3 + \frac{\partial u}{\partial \bar{x}_1}\frac{\partial v}{\partial \bar{x}_2} r_{12}\sigma_1\sigma_2 + \frac{\partial u}{\partial \bar{x}_2}\frac{\partial v}{\partial \bar{x}_3} r_{23}\sigma_2\sigma_3 + \frac{\partial u}{\partial \bar{x}_2}\frac{\partial v}{\partial \bar{x}_4}\sigma_2^2}{\sqrt{\left(\frac{\partial u}{\partial \bar{x}_1}\right)^2\sigma_1^2 + \left(\frac{\partial u}{\partial \bar{x}_2}\right)^2\sigma_2^2 + 2\frac{\partial u}{\partial \bar{x}_1}\frac{\partial u}{\partial \bar{x}_2} r_2\sigma_1\sigma_2} \times \sqrt{\left(\frac{\partial u}{\partial \bar{x}_3}\right)^2\sigma_3^2 + \left(\frac{\partial v}{\partial \bar{x}_2}\right)^2\sigma_2^2 + \frac{\partial u}{\partial \bar{x}_3}\frac{\partial v}{\partial \bar{x}_2} r_{32}\sigma_3\sigma_2}}$$

If the variable x_1, x_2, x_3 are mutually uncorrelated $r_{13}=0$, $r_{12}=0$, and $r_{23}=0$, and the correlation between the two functions $u = f(r_1, x_2)$, and $v = f(x_3, x_2)$ then reduces to

$$\frac{\frac{\partial u}{\partial \bar{x}_2}\frac{\partial v}{\partial \bar{x}_4}\sigma_2^2}{\sqrt{\left(\frac{\partial u}{\partial \bar{x}_1}\right)^2\sigma_1^2 + \left(\frac{\partial u}{\partial \bar{x}_2}\right)^2\sigma_2^2}\sqrt{\left(\frac{\partial v}{\partial \bar{x}_3}\right)^2\sigma_3^2 + \left(\frac{\partial v}{\partial \bar{x}_2}\right)^2\sigma_2^2}} \quad \ldots \quad (8)$$

This then is the formula for the spurious correlation between the two functions, for it is the amount of correlation which persists between the functions $u = f(x_1, x_2)$ and $v = f(x_3, x_2)$ even if x_1, x_2, x_3 be so chosen that they are mutually uncorrelated.

4. A special and very interesting case of this, and one which Professor Karl Pearson considered in his paper quoted at the beginning of this article arises when $u = \frac{x_1}{x_2}$ and $v = \frac{x_3}{x_2}$. The amount of spurious correlation between these indices is then equal to

$$\frac{\bar{x}_1 \bar{x}_3 \sigma_2^2}{\sqrt{\bar{x}_2^2 \sigma_1^2 + \bar{x}_1^2 \sigma_2^2} \sqrt{\bar{x}_2^2 \sigma_3^2 + \bar{x}_3^2 \sigma_2^2}}$$

$$= \frac{\sigma_2^2 / \bar{x}_2^2}{\sqrt{\frac{\sigma_1^2}{\bar{x}_1^2} + \frac{\sigma_2^2}{\bar{x}_2^2}} \sqrt{\frac{\sigma_3^2}{\bar{x}_3^2} + \frac{\sigma_2^2}{\bar{x}_2^2}}}$$

$$= \frac{V_2^2}{\sqrt{V_1^2 + V_2^2} \sqrt{V_3^2 + V_2^2}},$$

where V_1, V_2, V_3 are the co-efficients of variation of x_1, x_2, x_3, *i.e.*, the ratios of the standard deviations of these variables to their means. This result is the same as Professor Pearson's.

If the V's are all equal as in most organic measurements, the spurious correlation will be equal to ·5. Thus, there is the possibility of a biologist reading as high a correlation co-efficient as ·5 into his calculations when there is no correlation at all between the organs merely owing to the peculiarity of his arithmetic.

It is easy to show from our Lemma that if x and y are any two variables, the correlation between $\psi(x)$ and $\psi(y)$ is the same as the correlation between x and y (up to terms of the second order). It ofllows that if we replace $\frac{x_1}{x_2}$, $\frac{x_3}{x_2}$ by $\psi\left(\frac{x_1}{x_2}\right)$, $\psi\left(\frac{x_3}{x_2}\right)$ respectively the same spurious correlation of precisely the same amount will reappear.

AN ORDER-RESULT CONCERNING THE ϕ-FUNCTION

BY S. SIVASANKARANARAYANA PILLAI.

Let

$$S = \sum_{n} \phi(nv + c), \qquad [nv + c \leqslant x]$$

where $\phi(x)$ is the number of integers less than and prime to x. The object of this note is to prove the following

THEOREM:

$$S = \frac{3}{\pi^2}\, \theta_c(v) . x^2 + O(x \log x),$$

where

$$\theta_c(v) = \frac{\phi\{(v, c)\}}{(v, c)\, \phi(v)} \prod_{p/v} \frac{p}{p+1},$$

where (v, c) stands for the greatest common divisor of v and c.

To prove this theorem, we require two lemmas.

Lemma 1. The congruence

$$ny \equiv c \qquad (\text{mod. } v)$$

is solvable only when $d = (n, v)$ divides c, and then if y is a solution,

$$y + \frac{kv}{d} \qquad (k = 0, 1, \ldots\ d - 1)$$

are also solutions.

This is a well-known theorem.*

Lemma 2. When $1 \leqslant a_0 \leqslant t - 1$,

$$T = \sum_{\substack{r \\ a_0 + rt \leqslant m}} (a_0 + rt) = \frac{m^2}{2t} + O(m).$$

Let

$$l = \left[\frac{m - a_0}{t}\right].$$

* Matthews: "*Theory of numbers,*" pp. 10—12.

Then,

$$\sum_{r=0}^{l}(1 + rt) \leqslant T \leqslant \sum_{r=0}^{l}(t - 1 + rt);$$

that is,

$$\frac{tl^2}{2} + O(l) \leqslant T \leqslant \frac{tl^2}{2} + O(l).$$

Therefore

$$T = \frac{tl^2}{2} + O(l).$$

But

$$l = \frac{m}{t} + O(1).$$

Hence

$$T = \frac{m^2}{2t} + O(m).$$

Now we are in a position to prove the main theorem.

$$S = \sum_{n}(nv + c) \sum_{\delta/(nv+c)} \frac{\mu(\delta)}{\delta} \qquad (mv + c \leqslant x)$$

$$= \sum_{n-1}^{x} \frac{\mu(n)}{n} \sum nr \qquad \{ nr \leqslant x,\ nr \equiv c \ (\text{mod. } v) \}$$

$$= \sum_{n=1}^{x} \mu(n) \sum r \qquad \{ r \leqslant \frac{x}{n},\ nr \equiv c \ (\text{mod. } v) \}$$

$$= S_0 + \sum_{t=0}^{l} S_{p_t} + \sum_{\substack{t,\,s \\ t \leqslant l \\ s \leqslant l \\ t \neq s}} S_{p_t p_s} + \ldots + S_{p_1 p_2 \ldots p_l} + S'$$

where $(v, c) = p_1^{a_1} p_2^{a_2} \ldots p_l^{a_l}$;

and in S_0, $(n, v) = 1$

in S_{p_t}; $(n, v) = p_t$

in $S_{p_t p_s}$; $(n, v) = p_t p_s$

... ...

in $S_{p_1 \ldots p_l}$ $(n, v) = p_1 \ldots p_l$;

and in S', n runs through all other values.

From the fact that $\mu(n) = 0$, when n is divisible by a square and from the lemma 1, we get $S' = 0$.

In considering other sums, let us take any one, say

$S_{p_t p_s \cdots p_u}$: where $p_t p_s \cdots p_u$ is a divisor of (v, c).

$$S_{p_t p_s \cdots p_u} = \sum_{n \leqslant x}' \mu(n) \sum r \qquad \left\{ r \leqslant \frac{x}{n},\ nr \equiv c \ (\text{mod. } v) \right\}$$

where the accent indicates that $(n, v) = p_t p_s \cdots p_u$

Let a_0 be the smallest solution of the congruence

$$ny \equiv c \ (\text{mod. } v).$$

Then

$$\sum r \qquad \left\{ r \leqslant \frac{x}{n},\ nr \equiv c \ (\text{mod. } v),\ (n, v) = p_t p_s \cdots p_u \right\}$$

$$= a_0 + \left(a_0 + \frac{v}{p_t p_s \cdots p_u}\right) + \ldots + \left(a_0 + h \cdot \frac{v}{p_t p_s \cdots p_u}\right),$$

where $\quad a_0 + \dfrac{hv}{p_t p_s \cdots p_u} \leqslant \dfrac{x}{n}$;

$$= \frac{x^2}{n^2} \times \frac{1}{2v/(p_t p_s \cdots p_u)} + O\left(\frac{x}{n}\right), \qquad \text{from lemma 2,}$$

$$= \frac{p_t p_s \cdots p_u}{2v} \cdot \frac{x^2}{n^2} + O\left(\frac{x}{n}\right).$$

Therefore,

$$S_{p_t p_s \cdots p_u} = \sum_{n \leqslant x}' \mu(n) \left\{ \frac{p_t p_s \cdots p_u}{2v} \cdot \frac{x^2}{n^2} + O\left(\frac{x}{n}\right) \right\}$$

$$= \frac{p_t p_s \cdots p_u}{2v} \cdot x^2 \sum_{n \leqslant x}' \frac{\mu(n)}{n^2} + O\left(x \sum_{n \leqslant x} \frac{1}{n}\right)$$

$$= \frac{p_t p_s \cdots p_u}{2v}\, x^2 \left\{ \sum' \frac{\mu(n)}{n^2} + O\left(\frac{1}{x}\right) \right\} + O(x \log x)$$

$$= \frac{p_t p_s \cdots p_u}{2v}\, x^2 \sum' \frac{\mu(n)}{n^2} + O(x \log x). \qquad \ldots \quad (1)$$

Since the accent indicates $(n, v) = p_t\, p_s \ldots p_u$

$$\sum{}' \frac{\mu(n)}{n^2} = \frac{\mu(p_t\, p_s \ldots p_u)}{(p_t\, p_s \ldots p_u)^2} \sum{}'' \frac{\mu(n)}{n^2} \qquad \ldots \quad (2)$$

here the double accent indicates that $(n, v) = 1$.

Now,

$$\sum{}'' \frac{\mu(n)}{n^2} = \frac{\Pi\left(1-\frac{1}{p^2}\right)}{\underset{p/v}{\Pi}\left(1-\frac{1}{p^2}\right)} = \frac{6}{\pi^2} \frac{1}{\underset{p/v}{\Pi}\left(1-\frac{1}{p^2}\right)} \qquad \ldots \quad (3)$$

Therefore, from (1), (2) and (3)

$$S_{p_t\, p_s \ldots p_u} = \frac{3}{\pi^2} \cdot \frac{1}{v} \cdot \frac{1}{\underset{p/v}{\Pi}\left(1-\frac{1}{p^2}\right)} \cdot \frac{\mu(p_t\, p_s \ldots p_u)}{p_t\, p_s \ldots p_u} \cdot x^2$$
$$+ O(x \log x).$$

imilarly for the other sums.

Therefore,

$$S = \frac{3 \cdot 1}{\pi^2 v} \cdot \frac{x^2}{\underset{p/v}{\Pi}\left(1-\frac{1}{p^2}\right)} \left\{1 + \Sigma \frac{\mu(p_t)}{p_t} + \Sigma \frac{\mu(p_t\, p_s)}{p_t\, p_s} + \ldots\right\}$$
$$+ O(x \log x)$$

$$= \frac{3}{\pi^2 v} \times \frac{x^2}{\underset{p/v}{\Pi}\left(1-\frac{1}{p^2}\right)} \underset{p/(v,\, c)}{\Pi}\left(1-\frac{1}{p}\right) + O(x \log x)$$

$$= \frac{3}{\pi^2} \cdot \frac{\phi\{(v\ c)\}}{(v, c)\ \phi(v)\}} \underset{p/v}{\Pi}\left(\frac{p}{p+1}\right) \cdot x^2 + O(x \log x).$$

Thus, the proof is completed.

When $c = 0$,

$$S = \underset{v_n \leqslant x}{\Sigma_n}\ \phi(vn) = \frac{3}{\pi^2 \phi(v)} \underset{p/v}{\Pi}\left(\frac{p}{p+1}\right) \times x^2 + 0(x \log x)$$

ON SOME EXPANSIONS AND INTEGRALS INVOLVING THE PARABOLIC CYLINDER FUNCTIONS.*

By V. L. Mutatker,
Research Scholar, Allahabad.

In Art. 1 of the present paper the summation formula is found, in Art. 2, some expansions in terms of the parabolic cylinder functions are obtained, and in Art. 3 certain integrals involving $D_n(z)$ are evaluated.

1. Summation formulae.

The recurrence formula satisfied by $D_n(z)$ is

$$zD_n(z) = D_{n+1}(z) + nD_{n-1}(z). \qquad \dots \quad (1.1)$$

Hence,

$$yD_n(y) = D_{n+1}(y) + nD_{n-1}(y). \qquad \dots \quad (1.2)$$

Multiplying (1) by $D_n(y)$ and (2) by $D_n(z)$ and subtracting, we obtain

$$(z-y)\ D_n(z)\ D_n(y) = [\,D_{n+1}(z)\,D_n(y) - D_{n+1}(y)\,D_n(z)]$$
$$+ [\,D_{n-1}(z)\,D_n(y) - D_{n-1}(y)\,D_n(z)\,] \quad (1.3)$$

In (1.3) put $n = 1, 2, 3 \dots \dots n$ and multiply the resulting equations respectively by $1, \frac{1}{2!}, \frac{1}{3!} \dots \dots$ and add; we thus obtain

$$(z-y)\sum_{p=1}^{n} \frac{D_p(z)\,D_p(y)}{p!} = [\,D_0(z)\,D_1(y) - D_0(y)\,D_1(z)\,]$$
$$+ \frac{1}{n!}[\,D_{n+1}(z)\,D_n(y) - D_{n+1}(y)\,D_n(z)\,].$$

---

* I am indebted to Dr. P. L. Srivastava and Dr. Gorakh Prasad for their kind help and encouragement in the preparation of the paper,

But

$$(z-y)\ D_0(y)\ D_0(z) = D_0(y)\ D_1(z) - D_1(y)\ D_0(z).$$

Therefore we have

$$(z-y)\sum_{p=0}^{n} \frac{D_p(z)\ D_p(y)}{\Gamma(p+1)} = \frac{1}{n!}\left[D_{n+1}(z)\ D_n(y) - D_{n+1}(y)\ D_n(z) \right] \quad \ldots \quad (1.4)$$

If now we put $z = y + \varepsilon$ in (1.4), then

$$\varepsilon \sum_{p=0}^{n} \frac{D_p(y)\ D_p(y+\varepsilon)}{\Gamma(p+1)}$$

$$= \frac{1}{n!}\left[\varepsilon\left(D_n(y)\ D'_{n+1}(y) - D_{n+1}(y)\ D'_n(y)\right) + O(\varepsilon^2)\right].$$

Comparing the co-efficients of ε, we obtain

$$\sum_{p=0}^{n} \frac{[D_p(y)]^2}{\Gamma(p+1)} = \frac{1}{n!}\left[D_n(y)\ D'_{n+1}(y) - D_{n+1}(y)\ D'_n(y)\right]. \quad \ldots \quad (1.5)$$

2. Some expansions in terms of the Parabolic Cylinder Functions.

It is known* that

$$e^{zt-\frac{1}{2}t^2-\frac{1}{4}z^2} = \sum_{n=0}^{\infty} \frac{D_n(z)}{n!}\, t^n. \quad \ldots \quad (2.01)$$

Differentiate (2.01) with respect to t and then put $t = e^{\theta i}$

$$\left(z - e^{\theta i}\right)\ e^{ze^{\theta i} - \frac{1}{2}e^{2\theta i} - \frac{1}{4}z^2} = \sum_{n=0}^{\infty} \frac{n\ D_n(z)}{n!}\, e^{(n-1)\theta i}.$$

---

* Hari Shanker, *Proc. Benares Math. Soc.*, Vol. VI (1924). For a direct proof, see Gorakh Prasad, *Proc Benares Math Soc.*, Vol. VIII—VIII (1925—26), pp. 47-48.

Equating real and imaginary parts we have

$$e^{z\cos\theta - \frac{1}{2}\cos 2\theta - \frac{1}{4}z^2}\,[\,z\cos(z\sin\theta - \tfrac{1}{2}\sin 2\theta) - \cos(z\sin\theta - \tfrac{1}{2}\sin 2\theta + \theta)\,]$$

$$= \sum_0^\infty \frac{n\,D_n(z)}{n!}\cos(n-1)\,\theta. \qquad \ldots (2.02)$$

And

$$e^{z\cos\theta - \frac{1}{2}\cos 2\theta - \frac{1}{4}z^2}\,[\,z\sin(z\sin\theta - \tfrac{1}{2}\sin 2\theta) - \sin(z\sin\theta - \tfrac{1}{2}\sin 2\theta + \theta)\,]$$

$$= \sum_0^\infty \frac{n\,D_n(z)}{n!}\sin(n-1)\,\theta. \qquad \ldots (2.03)$$

Put $\theta = \frac{\pi}{2}$ in (2.02) and (2.03): then we obtain

$$(z\cos z + \sin z)\, e^{\frac{1}{2} - \frac{1}{4}z^2} = \sum_0^\infty (-1)^n \frac{D_{2n+1}(z)}{\Gamma(2n+1)}; \ldots (2.04)$$

$$(z\sin z - \cos z)\, e^{\frac{1}{2} - \frac{1}{4}z^2} = \sum_0^\infty (-1)^n \frac{D_{2n+2}(z)}{\Gamma(2n+2)}. \ldots (2.05)$$

Differentiate (2.02) and (2.03) with respect to θ and put $\theta = \pi/2$; then

$$\left[\, 2z\sin z + (z^2 - 2)\cos z \,\right] e^{\frac{1}{2} - \frac{1}{4}z^2} = \sum_0^\infty (-1)^n \frac{D_{2n+2}(z)}{(2n)!}: \ldots (2.06)$$

$$\left[\, (z^2 - 2)\sin z - 2z\cos z \,\right] e^{\frac{1}{2} - \frac{1}{4}z^2} = \sum_0^\infty (-1)^n \frac{D_{2n+3}(z)}{(2n+1)!}. \ldots (2.07)$$

Differentiate (2.03) twice with respect to θ and put $\theta = 0$. We have

$$z(z-2)\, e^{z - \frac{1}{2} - \frac{1}{4}z^2} = \sum_0^\infty \frac{D_{n+2}(z)}{n!}. \qquad \ldots (2.08)$$

Again (2.02) might be differentiated $2r$ times and (2.03) $2r+1$ times and then if we put $\theta = 0$ we shall get the sum of the series of the form

$$\sum_{n=0}^\infty \frac{n\,(n-1)^r D_n(z)}{n!}. \qquad \ldots (2.06)$$

Now expand each side of (2.03) in powers of sin θ. Equating the co-efficients of sin θ we get (2.03) again. Equating the co-efficients of $\sin^3\theta$ we obtain

$$e^{z-\frac{1}{2}-\frac{1}{4}z^2}\left[\frac{z^2-1}{2}-\frac{(z-1)^4}{6}-\frac{1}{2}z(z-2)^2\right]$$

$$=-\sum_{0}^{\infty}\frac{n\,D_n(z)}{n!}\cdot\frac{(n-1)\left[(n-1)^2-1\right]}{3!}$$

$$=-\sum_{0}^{\infty}\cdot\frac{D_n(z)}{n!}\,\frac{n^2(n-1)(n-2)}{3!}. \quad \dots (2.10)$$

Again by expanding each side of (2.02) in powers of sin θ and equating the constant terms, the co-efficients of $\sin^2\theta$ and $\sin^4\theta$, we obtain

$$(z-1)\,.\,e^{z-\frac{1}{2}-\frac{1}{4}z^2}=\sum_{0}^{\infty}\frac{D_n(z)}{(n-1)!};^{*} \quad \dots (2.11)$$

$$e^{z-\frac{1}{2}-\frac{1}{4}z^2}\left[\frac{1}{2}-\frac{(z-1)^3}{2}+(z-1)\left(2-\tfrac{1}{2}z\right)\right]$$

$$=-\sum_{0}^{\infty}\frac{D_n(z)}{n!}\cdot\frac{n(n-1)^2}{2!}; \quad \dots (2.12)$$

and

$$e^{z-\frac{1}{2}-\frac{1}{4}z^2}\left[\frac{(z-1)^5}{4!}-\frac{1}{12}(z-1)^3(8-3z)-\frac{3}{4}(z-1)^2\right.$$

$$\left.+\frac{1}{4}\left(1-\tfrac{1}{2}z\right)(z-1)(6-z)+\frac{1}{2}(2-z)\right]$$

$$=\sum_{0}^{\infty}\frac{D_n(z)}{n!}\cdot\frac{n(n-1)^2\left[(n-1)^2-2^2\right]}{4!}; \quad \dots (2.13)$$

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* This expansion has already been obtained by Dr. Gorakh Prasad, see *Proc. Benares Math. Soc.*, Vol. VII—VIII (1925—26), p. 50.

Differentiating (2.01) m times with respect to t we obtain

$$e^{zt-\frac{1}{2}t^2-\frac{1}{4}z^2}\left[\sum_{p=0}^{\substack{m/2 \text{ or}\\(m-1)/2}} \frac{\Gamma(m+1)}{2^p\,\Gamma(p+1)\,\Gamma(m-2p+1)}(z-t)^{m-2p}\right]$$

$$=\sum_{n=m}^{\infty}\frac{D_n(z)}{(n-m)!}t^{n-m} \qquad \dots (2.14)$$

$\frac{m}{2}$ or $\frac{m-1}{2}$ being taken according as m is even or odd.

Put $t = e^{\theta i}$ in (2.14) and separate into real and imaginary parts; then

$$e^{z\cos\theta-\frac{1}{2}\cos 2\theta-\frac{1}{4}z^2}\sum_{p=0}^{\substack{m/2 \text{ or}\\(m-1)/2}} \frac{\Gamma(m+1)}{2^p\,\Gamma(p+1)\,\Gamma(m-2p+1)}\times$$

$$\left[z^{m-2p} + {}_{m-2p}C_1 z^{m-2p-1}\cos(z\sin\theta-\tfrac{1}{2}\sin 2\theta+\theta)+\dots\dots\right.$$
$$\left.+\cos(z\sin\theta-\tfrac{1}{2}\sin 2\theta+(m-2p)\theta)\right]$$

$$=\sum_{n=m}^{\infty}\frac{D_n(z)}{(n-m)!}\cos(n-m)\theta; \qquad \dots (2.15)$$

and

$$e^{z\cos\theta-\frac{1}{2}\cos 2\theta-\frac{1}{4}z^2}\sum_{p=0}^{\substack{m/2 \text{ or}\\(m-1)/2}} \frac{\Gamma(m+1)}{2^p\,\Gamma(p+1)\,\Gamma(m-2p+1)}\times$$

$$\left[z^{m-2p} + {}_{m-2p}C_1 z^{m-2p-1}\sin(z\sin\theta-\tfrac{1}{2}\sin 2\theta+\theta)+\dots\dots\right.$$
$$\left.+\sin(z\sin\theta-\tfrac{1}{2}\sin 2\theta+(m-2p)\theta)\right]$$

$$=\sum_{n=m}^{\infty}\frac{D_n(z)}{(n-m)!}\sin(n-m)\theta \qquad \dots (2.16)$$

If we put $t = i$ in (2.14) we obtain,

$$e^{\frac{1}{2}-\frac{1}{4}z^2} \sum_{p=0}^{m/2 \text{ or } (m-1)/2} \frac{\Gamma(m+1)}{2^p\,\Gamma(p+1)\,\Gamma(m-2p+1)} \times$$

$$[(z^{m-2p} - {}_{m-2p}C_2\, z^{m-2p-2} + {}_{m-2p}C_4\, z^{m-2p-4} \ldots \ldots) \cos z$$

$$-({}_{m-2p}C_1\, z^{m-2p-1} - {}_{m-2p}C_3\, z^{m-2p-3} + \ldots \ldots) \sin z]$$

$$= \sum_{r=0}^{\infty} (-1)^r \cdot \frac{D_{m+2r}(z)}{(2r)!}. \qquad \ldots (2.17)$$

And

$$e^{\frac{1}{2}-\frac{1}{4}z^2} \sum_{p=0}^{m/2 \text{ or } (m-1)/2} \frac{\Gamma(m+1)}{2^p\,\Gamma(p+1)\,\Gamma(m-2p+1)} \times$$

$$[(z^{m-2p} - {}_{m-2p}C_2\, z^{m-2p-2} + {}_{m-2p}C_4\, z^{m-2p-4} \ldots \ldots) \sin z$$

$$+({}_{m-2p}C_1\, z^{m-2p-1} - {}_{m-2p}C_3\, z^{m-2p-3} + \ldots \ldots) \cos z]$$

$$= \sum_{r=0}^{\infty} (-1)^r \cdot \frac{D_{m+2r+1}(z)}{(2r+1)!}. \qquad \ldots (2.18)$$

Differentiate (2.16) once with respect to θ and then put $\theta = 0$; we have

$$e^{z-\frac{1}{2}-\frac{1}{4}z^2} \sum_{p=0}^{m/2 \text{ or } (m-1)/2} \frac{\Gamma(m+1)}{2^p\,\Gamma(p+1)\,\Gamma(m-2p+1)} \times$$

$$[{}_{m-2p}C_1\, z^{m-2p} + {}_{m-2p}C_2\, z^{m-2p-2}(z+1) + {}_{m-2p}C_3\, z^{m-2p-3}(z+2)$$

$$+ \ldots \ldots + (z+m-2p-1)]$$

$$= \sum_{n=m}^{\infty} \frac{D_n(z)}{\Gamma(n-m)}. \qquad \ldots (2.19)$$

It has been proved that*

$\Sigma\, a_n\,(n\,!)^{-1}\, D_n(z)$ is convergent if a_{n+1}/a_n remains finite. Thus all the expansions considered in this article are convergent.

3. Evaluation of certain definite integrals.

We have

$$\sum_{n=0}^{\infty} \frac{D_n(kz)}{n\,!}\, t^n = e^{kzt-\frac{1}{2}t^2-\frac{1}{4}k^2z^2}$$

$$= e^{-\frac{1}{4}z^2(k^2-1)} \,.\, e^{-\frac{1}{2}t^2(1-k^2)} \sum_{n=0}^{\infty} \frac{D_n(z)}{n!} k^n t^n$$

Equating the co-efficients of t^n we obtain

$$D_n(kz) = e^{-\frac{1}{4}z^2(k^2-1)}\, \Gamma(n+1) \times$$

$$\sum_{p=0}^{n/2 \text{ or } (n-1)/2} \frac{k^{n-2p}(1-k^2)^p(-1)^p}{\Gamma(n-2p+1)\,\Gamma(p+1)\,2_p} D_{n-2p}(z). \quad \ldots \quad (3.01)$$

Multiply both sides of (3.01) by $e^{\frac{1}{4}z^2(k^2-2)+bz}$ and integrate between the limits $-\infty$ and $+\infty$. We have

$$\int_{-\infty}^{\infty} e^{\frac{1}{4}z^2(k^2-2)+bz}\, D_n(kz)\, dz$$

$$= \Gamma(n+1) \sum_{p=0}^{n/2 \text{ or } (n-1)/2} \frac{k^{n-2p}(1-k^2)^p(-1)^p}{\Gamma(n-2p+1)\,\Gamma(p+1)\,2p} \times$$

$$\int_{-\infty}^{\infty} e^{bz-\frac{1}{4}z^2}\, D_{n-2p}(z)\, dz$$

$$= \sqrt{2\pi}\, e^{\frac{1}{2}b^2}\, \Gamma(n+1) \sum_{p=0}^{n/2 \text{ or } (n-1)/2} \frac{(-1)^p(kn)^{n-2p}(1-k^2)^p}{2^p\,\Gamma(p+1)\,\Gamma(n-2p+1)}$$

$$\ldots \quad (3.02)$$

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* Gorakh Prasad, *Proc. Benares Math. Soc.*, Vol. VII—VIII.

On employing the result*

$$\int_{-\infty}^{\infty} e^{zt-\frac{1}{4}z^2} D_n(z)\, dz = \sqrt{2\pi}\, t^n e^{\frac{1}{2}t^2}.$$

From (3.02) it is clear that

$$\int_{-\infty}^{\infty} e^{\frac{1}{4}z^2(k^2-2)} D_n(kz)\, dz$$

$$= \sqrt{2\pi}\, \Gamma(n+1) \frac{(-1)^{n/2}(1-k^2)^{n/2}}{\Gamma(n/2+1)\, 2^{n/2}} \quad \dots (3.03)$$

if n is even, and vanishes if n is odd.

3.1. To evaluate the integral

$$\int_{-\infty}^{\infty} e^{\frac{1}{4}z^2(k^2-1)} D_m(z)\, D_n(kz)\, dz,$$

multiply both sides of (3.01) by $D_m(z)$ and integrate between the limits $-\infty$ and $+\infty$; then

$$\int_{-\infty}^{\infty} e^{\frac{1}{4}z^2(k^2-1)} D_m(z)\, D_n(kz)\, dz$$

$$= \Gamma(n+1) \sum_{p=0}^{n/2 \text{ or } (n-1)/2} \frac{(-1)^p k^{n-2p}(1-k^2)^p}{\Gamma(n-2p+1)\, \Gamma(p+1)\, 2^p} \int_{-\infty}^{\infty} D_m(z)\, D_{n-2p}(z)\, dz.$$

Case 1. If $m < n$

$$\int_{-\infty}^{\infty} e^{\frac{1}{4}z^2(k^2-1)} D_m(z)\, D_n(kz)\, dz = 0; \quad \dots (3.11)$$

Case 2. If $m < n$, the integral vanishes except when

$$n - m = 2p \qquad \left(p = 0, 1, 2, \dots \dots \frac{n}{2} \text{ or } \frac{n-1}{2}\right).$$

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* Gorakh Prasad, *Proc. Benares Math. Soc.*, Vol. VII—VIII, 1925-26, p. 49.

In the exceptional case the value is

$$\int_{-\infty}^{\infty} e^{\frac{1}{4} z^2 (k^2-1)} D_m(z) D_n(kz)\, dz$$

$$= \sqrt{2\pi}\ \Gamma(n+1) \frac{k^{n-2p} (1-k^2)^p (-1)^p}{\Gamma(p+1)\, 2^p} \quad \dots (3.12)$$

3.2. With the help of (3.11) and (3.12) we can obtain the value of the definite integral

$$\int_{-\infty}^{\infty} e^{\frac{1}{4} z^2 (k^2-1)} D_m(z)\, D_n(z)\, D_p(kz)\, dz.$$

This is zero if $m+n+p$ is odd; if $m+n+p$ is even, we have by virtue of the two series,*

$$D_m(z)\, D_n(z) = \sum_{p=0}^{\infty} C_p\, D_{2p}(z) \quad [m+n \text{ even and } m \not> n],$$

where,

$$C_p = \sum_{r=0}^{m} \frac{m!\, n!}{r!\,(m-r)!\,(n-r)!}\ n+m-2, p$$

and

$$D_m(z)\, D_n(z) = \sum_{p=0}^{\infty} d_p\, D_{2p+1}(z) \quad [m+n \text{ odd and } m \not> n]$$

where $$d_p = \sum_{r=0}^{m} \frac{m!\, n!}{r!\,(m-r)!\,(n-r)!}\ n+m-2r^b, r$$

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* Gorakh Prasad, *Proc. Benares Math. Soc.*, Vol. II, p. 21 (1920).

the relation

$$\int_{-\infty}^{\infty} e^{\frac{1}{2}z^2(k^2-1)} D_m(z) D_n(z) D_p(kz)\, dz$$

$$= \sqrt{2\pi}\, \Gamma(p+1) \sum_{r=0}^{p/2} \frac{(-1)^r\, c_{p/2-r}\, k^{p-2r}}{\Gamma(r+1)} \frac{(1-k^2)^r}{2^r}$$

if $m+n$ is even;

and has the value

$$\sqrt{2\pi}\, \Gamma(p+1) \sum_{r=0}^{(p-1)/2} \frac{(-1)^r\, d_{(p-1)/2-r}\, k^{p-2r}}{\Gamma(r+1)} \frac{(1-k^2)^r}{2^r}$$

if $m+n$ is odd

3.3. The integral $\int_{-\infty}^{\infty} e^{\frac{1}{2}z^2(k^2-2)} \begin{matrix}\sin mz\\ \cos mz\end{matrix}\, D_n(kz)\, dz.$

Putting $t = im$ in (2.01) and separating into real and imaginary parts, we have

$$e^{-\frac{1}{2}z^2} \cos mz = e^{-\frac{1}{2}m^2} \sum_{n=0}^{\infty}{}' (-1)^n \frac{(m)^{2n}}{(2n)!} D_{2n}(z) \quad \ldots \quad (3.31)$$

and

$$e^{-\frac{1}{2}z^2} \sin mz = e^{-\frac{1}{2}m^2} \sum_{n=0}^{\infty} (-1)^n \frac{(m)^{2n+1}}{(2n+1)!} D_{2n+1}(z). \quad (3.32)$$

Multiply both sides of (3.31) by $e^{\frac{1}{2}z^2(k^2-1)} D_{2b}(kz)$ and integrate between the limits $-\infty$ and $+\infty$; we obtain,

$$\int_{-\infty}^{\infty} e^{\frac{1}{2}z^2(k^2-2)} \cos mz\, D_{2b}(kz)\, dz$$

$$= e^{-\frac{1}{2}m^2} \sum_{n=0}^{\infty} (-1)^n \frac{m^{2n}}{(2n!} \int_{-8}^{\infty} e^{\frac{1}{2}z^2(k^2-1)} D_{2n}(z) D_{2b}(kz)\, dz.$$

With the help of (3.11) and (3.12), it becomes

$$\int_{-\infty}^{\infty} e^{\frac{1}{4}z^2(k^2-2)} \cos mz \, D_{2b}(kz) \, Pz$$

$$= \sqrt{2\pi}\; e^{-\frac{1}{2}m^2} \Gamma(2b+1) \sum_{p=0}^{b} (-1)^b \frac{(km)^{2b-2p}(1-k^2)^p}{\Gamma(2b-2p+1)\,\Gamma(p+1)\,2^p} \quad \ldots \quad (3.33)$$

In a similar manner, with the help of (3.11) and (3.12) we obtain the following integrals:

$$\int_{-\infty}^{\infty} e^{\frac{1}{4}z^2(k^2-2)} \sin mz \, D_{2b+1}(kz)\, dz$$

$$= \sqrt{2\pi}\; e^{-\frac{1}{2}m^2} \Gamma(2b+2) \sum_{p=0}^{b} (-1)^b \frac{(km)^{2b-2p+1}(1-k^2)^p}{\Gamma(2b-2p+2)\,\Gamma(p+1)\,2^p} \quad \ldots \quad (3.34)$$

and

$$\int_{-\infty}^{\infty} e^{\frac{1}{4}z^2(k^2-2)} \cos mz \, D_{2b+1}(kz)\, dx$$

$$= \int_{-\infty}^{\infty} e^{\frac{1}{4}z^2(k^2-2)} \sin mz \, D_{2b}(kz)\, dz = 0 \ldots \quad (3.35)$$

3.4. The Integral

$$\int_0^{\infty} \frac{e^{\frac{1}{4}\cot^2 x\,.\,(k^2-2)}\, D_{2n}(k \cot x)}{(a^2+x^2)}\, dx.$$

In (3.01) replace z by $\cot x$. Multiply both the sides by

$$\frac{e^{\frac{1}{4}\cot^2 x\,.\,(k^2-2)}}{a^2+x^2}$$

integrate between the limits 0 and ∞; then we obtain

$$\int_0^{\infty} \frac{e^{\frac{1}{4}\cot^2 x\,.\,(k^2-2)}\, D_{2n}(k \cot x)}{(a^2+x^2)}\, dx$$

$$= \Gamma(2n+1) \sum_{p=0}^{n} \frac{k^{2n-2p}\,(1-k^2)^p\,(-1)^p}{\Gamma(2n-2p+1)\,\Gamma(p+1)\,2^p} \times$$

$$\int_0^\infty \frac{e^{-\frac{1}{4}\cot^2 x}\, D_{2n-2p}(\cot x)}{a^2+x^2}\,dx$$

$$= \Gamma(2n+1)\, a^{-1} \left(\frac{\pi}{2}\right)^{1/2} e^{\frac{1}{4}\coth^2 a} \times$$

$$\sum_{p=0}^{n} (-1)^n \frac{k^{2n-2p}\,(1-k^2)^p\, D_{2p-2n-1}(\coth a)}{\Gamma(p+1)\,2^p}. \qquad \dots (3.41)$$

Since*

$$\int_0^\infty \frac{e^{-\frac{1}{4}\cot^2 x}\,.\,D_{2n}(\cot x)}{a^2+x^2}$$

$$= (-1)^n\, a^{-1} \left(\frac{\pi}{2}\right)^{1/2} \Gamma(2n+1)\, e^{\frac{1}{4}\coth^2 a}\, D_{-2n-1}(\coth a).$$

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* R. S. Varma, *Tohoku Math. Journal* (June 1931), p. 106.

THE THEORY OF BILINEAR AND DOUBLE-BINARY FORMS

By R. Vaidyanathaswamy.

I. Let

$$F = \sum a_{ij} x_i y_j \qquad (i, j = 1, 2, \ldots, n)$$

be a bilinear form in two sets of digredient variables $(x_1, x_2, \ldots x_n)$, $(y_1, y_2, \ldots y_n)$. The matrix $| a_{ij} |$ of order n may be called the matrix of F, and its rank r, the rank of F. In matrix notation, F may be written:

$$F = | x_1 \; x_2 \ldots x_n | \times \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & \ldots & \ldots & a_{nn} \end{vmatrix} \times \begin{vmatrix} y_1 \\ y_2 \\ \ldots \\ y_n \end{vmatrix}$$

Now from theory of matrices, it is known that there exist two non-singular square matrices S, T, such that

$$S . | a_{ij} | . T = [1]_r,$$

where $[1]_r$ denotes a matrix whose only non-vanishing elements are r units in the principal diagonal. Writing

$$T^{-1} . \begin{vmatrix} y_1 \\ y_2 \\ \ldots \\ \ldots \end{vmatrix} = \begin{vmatrix} y'_1 \\ y'_2 \\ \ldots \\ y'_n \end{vmatrix}; \text{ and } | x_1 \; x_2 \ldots x_n | \; S^{-1} = | x_1' \; x_2' \ldots x_n' | ;$$

we have

$$F = | x_1' \; x_2' \ldots x_n' | . [1]_r . \begin{vmatrix} y_1' \\ \ldots \\ y_r' \end{vmatrix} = x_1' y_1' + \ldots x_r' y_r'.$$

This is the reduction of F to the canonical form. The variables $x_1' \ldots x_r'$ as well as $y_1' \ldots y_r'$ are linearly independent, since the matrices S, T are non-singular.

That a corresponding theory holds for forms in two binary variables was shown by the writer in the paper 'On the rank of the double-binary form' *Proc, Lond Math. Soc.*, Vol. 24, Part II, 1924).

Thus, if $x_1 \; x_2 \dots x_n$ are replaced $x^{n-1}, x^{n-2}, \dots, 1$, and $y_1, y_2, \dots y_n$ by $y^{n-1}, y^{n-2}, \dots, 1$, F becomes a double polynomial F (x, y), of orders $(n-1, n-1)$. The above reduction shews that F (x, y), supposed of rank r, can be written in the canonical shape:

$$f_1(x)\,\phi_1(y) + \dots + f_r(x)\,\phi_r(y),$$

where $(f_1, f_2, \dots f_r)$, and $(\phi_1, \phi_2, \dots, \phi_r)$

are two linearly independent sets of polynomials of order n—1.

II. Let now F be a symmetric bilinear form in co-gredient sets $(x_1, x_2, \dots, x_n)$, $(y_1, y_2, \dots, y_n)$, of rank r, and with *real* co-efficients a_{ij}. It is known from the theory of matrices that we can in this case find a real non-singular matrix S, such that (S′ being the transposed matrix of S),

$$S' . \mid a_{ij} \mid . S = [\pm 1]_r.$$

Hence as before, F can be reduced by a real co-gredient transformation of the variables, to the form :

$$F \equiv \varepsilon_1\, x_1'\, y_1' + \varepsilon_2\, x_2'\, y_2' + \dots + \varepsilon_r\, x_r'\, y'_r,$$

where each ε is $\pm$ 1. If the two sets of variables are not merely co-gredient, but identical, F becomes a real quadratic form of rank r, and the canonical form is :

$$F \equiv \varepsilon_1\, x'^2_1 + \dots + \varepsilon_r\, x'^2_r. \qquad \dots \quad (1)$$

We shall now suppose that the number s of negative ε's in this is not greater than the number $r-s$, of positive ε's (if this is not the case, it can be brought about by changing the sign of F). Then the ***signature*** $s \leqslant \frac{1}{2} r$. By writing

$$X'^2_p - X'^2_q = (X'_p + X'_q)(X'_p - X'_q),$$

we can now write F (supposed to have the signature $s \leqslant \frac{1}{2}r$) in the form :

$$F \equiv X_1^2 + \dots + X_{r-2s}^2 + Y_1 Y_1' + Y_2 Y_2' + \dots + Y_s Y_s', \qquad \dots \quad (2)$$

where the r linear forms $X_1\ X_2 \ldots X_{r-2s}, Y_1, Y_1', \ldots Y_s, Y_s'$ are independent. In this canonical shape, there occur $r-2s$ squared terms, and s product terms, all with the positive sign. We shall now interpret the signature of F, by means of the maximum order of real imbedded solutions of $F=0$.

A *solution* of $F=0$ is a set of real values of the n variables which satisfy the equation. An *imbedded solution of order k* of $F=0$ is a set of k independent solutions, so chosen that every real linear combination of them is also a solution. The theorem in question is, that if F has rank r and signature s ($\leqslant \frac{1}{2} r$), the maximum order of its imbedded solutions is $n-r+s$. It is clear from (2) that F possesses imbedded solutions of this order, namely solutions of the form:

$$X_1 = X_2 = \ldots = X_{r-2s} = Y_1 = Y_2 = \ldots = Y_s = 0.$$

Conversely, suppose F is known to possess an imbedded solution of order $n-t$, $X_1 = X_2 = \ldots = X_t = 0$. We may shew that $s \geqslant r-t$. For F must be of the shape

$$X_1Y_1 + X_2Y_2 + \ldots + X_tY_t,$$

where the t X's and $r-t$ of the Y's form a linearly independent set. If these Y's be $Y_1\ Y_2 \ldots Y_{r-t}$, it is easy by substituting linear expressions in $X_1, X_2 \ldots X_t, Y_1 \ldots Y_{r-t}$ for the remaining Y's to see that F can be written as a sum of $r-t$ products and a quadratic form in $2t-r$ variables. Thus $s \geqslant r-t$, and therefore the maximum order of imbedded solutions is $n-r+s$.

III. We now proceed to obtain the theory parallel to the above, for double-polynomials. If F is a real, symmetric polynomial of order $n-1$ in the two variables x, y, and of rank r, it follows at once from the above that it can be reduced to the canonical shape:

$$F(x, y) \equiv \varepsilon_1 f_1(x)\, f_1(y) + \varepsilon_2 f_2(x)\, f_2(y) + \ldots + \varepsilon_r f_r(x)\, f_r(y),$$

where the ε's are ± 1, and $f_1, f_2, \ldots, f_r$ are r linearly independent *real* polynomials. We assume as before that the signature s, that is, the number of negative ε's, is equal to or less than $\frac{1}{2}r$. We shall obtain

the interpretation of the signature s of $F(x, y)$ in the following form. The real, symmetric double-polynomial F will be said to be ***apolar*** to the polynomial ϕ, if $F(x, y)$ is apolar to $\phi(x)\,\phi(y)$; F will be said to be ***apolar*** to the linear system $\lambda_1\phi_1 + \ldots + \lambda_k\phi_k$, of order k (where ϕ's are real polynomials, and λ's are real parameters) if it is apolar to each polynomial of the system. We shall shew that the maximum order of linear systems apolar to F is $n - r + s$

The proof of this is reduced to the theory of real quadratic forms by the following theorem.

> ***If $F(x, y)$ is a real symmetric $(n-1, n-1)$ polynomial in x, y, of rank r and signature s $(\leqslant \frac{1}{2}\cdot)$, and if the real polynomial***
>
> $$\phi(x) = X_1 x^{n-1} - \left(\frac{n-1}{1}\right) X_2 x^{n-2} + \ldots,$$
>
> ***then the condition of apolarity of $F(x, y)$ and $\phi(x)\,\phi(y)$ is the vanishing of a quadratic form in $X_1, X_2, \ldots, X_n$, which has the same rank and the same signature as $F(x, y)$.***

For, if $F(x, y) = \Sigma\, a_{ij} x^{i-1} y^{j-1}$ $(i, j = 1, 2, \ldots, n;\ a_{ij} = a_{ji})$, then the condition of apolarity of $F(x, y)$ to $\phi(x)\,\phi(y)$ is:

$$\Sigma\, a_{ij} X_i X_j = 0.$$

Since the matrix of this quadratic from is identical with the matrix of $F(x, y)$, the theorem follows.

Hence from the proved fact that the maximum order of imbedded solutions of a real quadratic form is $n - r + s$, we deduce that the same expression gives the maximum order of apolar linear systems of real polynomials, possessed by F.

IV. We now pass on to the theory of the signature of Hermitian forms in $x_1, x_2, \ldots, x_n$ and their complex-conjugate variables, $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$ which are the natural generalisation of real quadratic forms. Taking

$$H(x) = \Sigma\, a_{ij} x_i \bar{x}_j \qquad (i, j = 1, 2, \ldots n:\ a_{ij} = \bar{a}_{ji}).$$

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NOTES AND QUESTIONS.

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Notes and Questions.

A Contribution to the inversion geometry of the bicircular quartic.*

1. It is well known that every bicircular quartic is self-inversive with respect to four mutually orthogonal circles.† These four circles determine three mutually harmonic involutions and the quartic contains an infinity of syzygetic tetrads defined by the three involutions.‡ It is the object of this note to find when the cross-ratio of the parameters of any four tetrads will be real; in other words, representing the parameter by a point in the Gauss plane, when the locus of this point will be a circle. It is found that the quartic should be sub-rational (*i.e.*), it should contain the cross-pair of every pair lying on it and belonging to one of the three involutions.§ Further, if the parameters of the perfect squares of the syzygetic pencil correspond to the points λ_1, λ_2, λ_3 of the Gauss plane, the circles corresponding to the sub-rational quartics belong to one of the three co-axal systems of circles whose limit points are $\lambda_1 \lambda_2$, $\lambda_2 \lambda_3$, and $\lambda_3 \lambda_1$ respectively.

At the outset it may be remarked that since the problem is invariant for inversion, it is enough if it is taken up in any special case arising as a result of any number of inversions. So the system of four mutually orthogonal circles can be inverted into the X-axis, the Y-axis, and the

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* I express my deep gratitude to Dr. Vaidyanathaswamy for kindly suggesting this problem to me.

† Ganguli, *Theory of plane curves*, Vol. II, page 107.

‡ A syzygetic tetrad is one which transforms into itself by each of the three mutually harmonic involutions.

§ The cross-pair of $A_1 A_2$ is the other pair of points of intersection of the circular lines through A_1 and A_2.

circles

$$x^2 + y^2 = 1, \text{ and } x^2 + y^2 = -1.$$

• Now the syzygetic tetrads are given by

$$z^4 - 2\lambda z^2 + 1 = 0$$

where $z = x + + iy$. The three mutually harmonic involutions are

$$I_1(z) = -z; \quad I_2(z) = \frac{1}{z}; \quad I_3(z) = -\frac{1}{z}.$$

2. The cartesian equation to a bicircular quartic invariant for all these involutions can be easily found to be

$$(x^2 + y^2)^2 + ax^2 + by^2 + 1 = 0$$

or

$$z^2 \bar{z}^2 + A(z^2 + \bar{z}^2) + B z\bar{z} + 1 = 0$$

where $z = x + iy$, $\bar{z} = x - iy$, while A and B are real. Corresponding to any point $A_0(z)$ on it, there are three involution pairs on it

$$A_0 A_1 (z, -z); \qquad A_0 A_2 \left(z, \frac{1}{z}\right); \qquad A_0 A_3 \left(z, -\frac{1}{z}\right)$$

belonging respectively to the involutions I_1, I_2, and I_3.

The cross pair of $A_0 A_1$ is given by

$$z_1 = z, \ \bar{z}_1 = -\bar{z} \text{ and } z_2 = -z, \ \bar{z}_2 = \bar{z}.$$

The quartic contains this if B = 0 and will then contain the cross pair of every pair of this involution I_1 lying on it. So we have a pencil of co-director sub-rational bicircular quartics given by

$$z^2 \bar{z}^2 + A(z^2 + \bar{z}^2) + 1 = 0. \qquad \dots \quad (1)$$

Similarly corresponding to the other two involutions, we have two other pencils of such quartics, whose equations are readily seen to be

$$z^2 \bar{z}^2 + (z^2 + \bar{z}^2) + B z\bar{z} + 1 = 0 \qquad \dots \quad (2)$$

$$z^2 \bar{z}^2 - (z^2 + \bar{z}^2) + B z\bar{z} + 1 = 0. \qquad \dots \quad (3)$$

3. Let us start from a syzygetic pencil

$$z^4 - 2\lambda z^2 + 1 = 0.$$

Suppose λ moves in a circle

$$4\lambda\bar{\lambda} + 4a\lambda + 4\bar{a}\bar{\lambda} + 4b = 0.$$

The locus of z is

$$\left(z^2 + \frac{1}{z^2}\right)\left(\bar{z}^2 + \frac{1}{\bar{z}^2}\right) + 2a\left(z^2 + \frac{1}{z^2}\right) + 2\bar{a}\left(\bar{z}^2 + \frac{1}{\bar{z}^2}\right) + 4b = 0$$

or

$$z^4\bar{z}^4 + 2az^4\bar{z}^2 + 2\bar{a}z^2\bar{z}^4 + z^4 + \bar{z}^4 + 4bz^2\bar{z}^2 - 2\bar{a}z^2 + 2a\bar{z}^2 + 1 = 0.$$

The locus is in general an 8-ic with 4-ple points at the circular points at infinity.

If this is to break up into two bicircular quartics, each having an infinity of tetrads of the above pencil, the above equation should break up into two factors of the type

$$z^2\bar{z}^2 + A(z^2 + \bar{z}^2) + B z\bar{z} + 1 = 0,$$

and $$z^2\bar{z}^2 + A'(z^2 + \bar{z}^2) + B' z\bar{z} + 1 = 0,$$

where A, B, A′, B′ are all real.

Identifying co-efficients we have

$$AA' = 1,\quad A + 1/A = 2a = 2\bar{a},\quad 2AA' + BB' + 2 = 4b,$$

$$AB' + A'B = 0,\ B + B' = 0.$$

From these we have three cases.

Case i. $B = B' = 0$, $A' = 1/A$, $A + 1/A = 2a$, $b = 1$ and a is real.

Hence the circle is

$$\lambda\bar{\lambda} + a(\lambda + \bar{\lambda}) + 1 = 0,$$

or in cartesians $$x^2 + y^2 + 2ax + 1 = 0;$$

and the two corresponding quartics are

$$z^2\bar{z}^2 + A(z^2 + \bar{z}^2) + 1 = 0,$$

$$z^2\bar{z}^2 + \frac{1}{A}(z^2 + \bar{z}^2) + 1 = 0,$$

the two belonging to a pencil of sub-rational quartics of the first type given in § 2. Further corresponding to this, we have a system of co-axal circles whose limit points are given by

$$\lambda = 1 \text{ and } \lambda = -1.$$

Case ii. $A = A' = 1$, $B = -B'$, $B^2 = 4(1-b)$, $a = 1$.

Here the cartesian equation of the circle is

$$x^2 + y^2 + 2x + b = 0,$$

and the corresponding quartics are

$$z^2\bar{z}^2 + z^2 + \bar{z}^2 + 1 + Bz\bar{z} = 0,$$

and $$z^2\bar{z}^2 + z^2 + \bar{z}^2 + 1 - Bz\bar{z} = 0,$$

which belong to the type (2) of § 2.

Case iii. $A = A' = -1$, $B = -B'$, $B^2 = 4(1-b)$, $a = -1$.

The equation to the circle is now

$$x^2 + y^2 - 2x + b = 0.$$

The corresponding quartics are

$$z^2\bar{z}^2 - (z^2 + \bar{z}^2) + 1 \pm Bz\bar{z} = 0,$$

Now the parameters of the perfect squares of the syzygetic pencil are $1, -1, \infty$. So we have finally the following:—

In order that the locus of the parameter of the syzygetic tetrads on a bicircular quartic curve should be a circle, the quartic should be sub-rational. Corresponding to the three pencils of co-director sub-rational bicircular quartics we have three systems of co-axal circles, whose limit points are (λ_1, λ_2), (λ_2, λ_3), (λ_3, λ_1) respectively, the parameters of the perfect squares of the syzygetic pencil.

These results can be shown to be true even in the special case where the locus of λ is a straight line.

NOTE. If one of the two associated quartics is real, the other is imaginary. For example in (1), the second contains the cross pairs of every A_0A_2, A_0A_3 of the first.

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On the Identical relation between the line-co-ordinates in a plane.

1. If p, q, r are the lengths of the perpendiculars from the vertices A, B, C of a triangle on any straight line, it is well known that there exists between them the identical relation

$$\Sigma a^2 (p-q)(p-r) = 4\Delta^2,$$

where Δ represents the area of the triangle ABC. It is this relation which is discussed here in this note.

THEOREM: *If ABC, A′B′C′ be two triangles inscribed in a circle such that AA′, BB′, CC′ are parallel to one another, then*

$$\Delta\, A'BC + \Delta\, B'CA + \Delta\, C'AB = \Delta\, ABC,$$

the areas being taken with their proper signs.

The proof of this result is immediate. For,

$$\Delta A'CA \equiv -\Delta AC'A', \quad \Delta AB'C \equiv -\Delta A'BC';$$

so that the quadrilaterals AA′CB′, AA′BC′ are equal in area.

Hence $\Delta A'CA + \Delta ACB' = \Delta AA'B + \Delta ABC' \qquad \ldots (1)$

But $\Delta A'BC + \Delta A'CA + \Delta A'AB = \Delta ABC \qquad \ldots (2)$

Subtracting (1) from (2) and taking the signs of the areas into account, we obtain

$$\Delta A'BC + \Delta B'CA + \Delta C'AB = \Delta ABC.$$

Now let p, q, r be the distances of A, B, C from a straight line parallel to AA′, BB′, or CC′ ; and let R be the circum-radius of ABC.

Then
$$A'B.AB = 2R(p-q),$$
$$A'C.AC = 2R(p-r).$$

$\therefore$
$$4\,\Delta A'BC\,.\,\Delta ABC = A'B.AB,\ A'C.AC \sin^2 A$$
$$= 4R^2(p-q)(p-r)\sin^2 A$$
$$= a^2(p-q)(p-r).$$

Similarly, $4\Delta B'CA.\Delta ABC = b^2(q-p)(q-r);$

and $4\Delta C'AB\,\Delta ABC = c^2(r-p)(r-q).$

$\therefore$

$$4(\Delta A'BC + \Delta B'CA + \Delta C'AB)\,\Delta ABC = \Sigma a^2(p-q)(p-r)$$

i.e.,
$$4\Delta^2 = \Sigma a^2(p-q)(p-r).$$

2. The following is an alternative method of deriving the above identity, which also gives incidentally the condition for perpendicularity of two straight lines in areal co-ordinates.

Let (p, q, r), (p_1, q_1, r_1) be the perpendiculars from the vertices A, B, C of the triangle of reference on the two perpendicular lines

$$px + qy + rz = 0,$$
$$p_1x + q_1y + r_1z = 0.$$

Taking these lines as the x- and y-axes of a rectangular cartesian frame, the co-ordinates of A, B, C are (p_1, p), (q_1, q), (r_1, r).

Let us write l, m, n, l_1, m_1, n_1 for $q - r, r - p, p - q, q_1 - r_1, r_1 - p_1, p_1 - q_1$ respectively.

We now have the following relations:

$$l + m + n = l_1 + m_1 + n_1 = 0 \qquad \ldots \quad (1)$$

$$4\,\Delta^2 = \begin{vmatrix} p_1 & p & 1 \\ q_1 & q & 1 \\ r_1 & r & 1 \end{vmatrix}^2 = \begin{vmatrix} p_1 - q_1 & p - q \\ q_1 - r_1 & q - r \end{vmatrix}^2 = (n_1 l - n l_1)^2.$$

Similarly, $\quad 4\,\Delta^2 = (m_1 l - l_1 m)^2 = (m_1 n - m n_1)^2. \qquad \ldots \; (2)$

Again, $\Sigma\, l_1^2\, mn = -(l_1^2 m + m_1^2\, l)(l + m) + lm\,(l_1 + m_1)^2$ from (1)

$$= -(l^2\, m_1^2 + m^2\, l_1^2 - 2\, lm\, l_1\, m_1)$$
$$= -(lm_1 - l_1 m)^2 \qquad \ldots \; (3)$$

Similarly $\Sigma\, l^2 m_1\, n_1 = -(lm_1 - l_1 m)^2.$

$$\Sigma\, ll_1\,(mn + m_1 n_1) = lmn\,\Sigma\, l_1 + l_1 m_1 n_1\,\Sigma\, l = 0 \qquad \ldots \; (4)$$

Also $\quad a^2 = l_1^2 + l^2,\; b^2 = m_1^2 + m^2,\; c^2 = n_1^2 + n^2$

and $\quad 2\,bc \cos A = b^2 + c^2 - a^2$

$$= m_1^2 + n_1^2 - (m_1 + n_1)^2 + m^2 + n^2 - (m + n)^2$$
$$= -2\,(m_1\, n_1 + mn),$$

i.e., $\quad bc \cos A = -(mn + m_1 n_1).$

Similarly $\quad ca \cos B = -(nl + n_1\, l_1),$

and $\quad ab \cos C = -(lm + l_1\, m_1). \qquad \ldots \; (5)$

These are the six equivalent conditions for perpendicularity.

$$\text{Now } \Sigma a^2 (p-q)(p-r) = -\Sigma (l^2 + l_1^2)\, mn$$

$$= -lmn \, \Sigma l - \Sigma l_1^2 \, mn$$

$$= -(lm_1 - l_1 m)^2 \text{ by (3) above}$$

$$= 4 \triangle^2. \qquad \ldots \quad (6)$$

From (4) and (5) we derive

$$\Sigma (q-r)(q_1 - r_1)\, bc \cos A = 0 \qquad \ldots \quad (7)$$

which is the well-known condition for the perpendicularity of the two lines $px + qy + rz = 0$ and $p_1x + q_1y + r_1z = 0$.

If these lines, instead of being perpendicular, be inclined to each other at an angle ω, we have

$$2\triangle = \begin{vmatrix} p_1 & p & 1 \\ q_1 & q & 1 \\ r_1 & r & 1 \end{vmatrix} \text{cosec } \omega.$$

$$\therefore \quad 4\triangle^2 \sin^2 \omega = (l_1 m - l m_1)^2, \text{ etc.}$$

when $\omega = 0$, we get the condition for parallelism

$$\frac{l}{l_1} = \frac{m}{m_1} = \frac{n}{n_1} \qquad \ldots \quad (8)$$

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On the order of the expression $1^1 . 2^2 . 3^3 \dots n^n$.

It is well known that the order of the expression

$$1 . 2 . 3 \dots n = n!$$

is $n^{n+\frac{1}{2}} e^{-n}$ and that

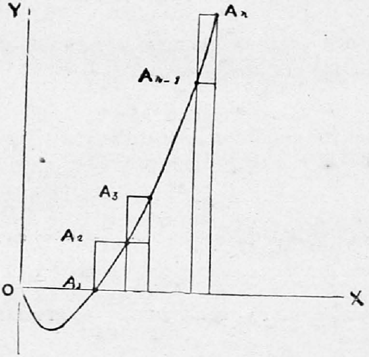
$$\frac{n!}{n^{n+\frac{1}{2}} e^{-n}} \to \sqrt{2\pi} \text{ as } n \to \infty.$$

In this paper it is shown that

$$\frac{1^1 . 2^2 \; 3^3 \dots n^n}{n^{an^2+bn+c} \, e^{-(b'n^2+a'n)}} \to \alpha$$

as $n \to \infty$ (where α is a finite number greater than zero) for some suitable values of a, b, c, a', b'.

Draw the curve $y = x \log x$ and take points A_1, A_2, ... A_n on the curve, the abscissae of these points being 1, 2, 3, ... n, respectively.



From the figure we can immediately deduce that

$$(2 \log 2 + 3 \log 3 + \dots + n \log n) > \int_1^n x \log x \, dx$$

$$> 2 \log 2 + 3 \log 3 + \dots + (n-1) \log (n-1)$$

$$\text{or} \int_1^{n+1} x \log x \, dx > 2 \log 2 + 3 \log 3 + \ldots + n \log n > \int_1^n x \log x \, dx$$

$$\text{or } (n+1)^{\frac{(n+1)^2}{2}} e^{-\frac{n(n+2)}{4}} > 1^1 . 2^2 . 3^3 \ldots n^n > n^{\frac{n^2}{2}} e^{-\frac{n^2-1}{4}}.$$

This suggests that $\dfrac{1^1 . 2^2 . 3^3 \ldots n^n}{n^{an^2+bn+c} \, e^{-(a'n^2+b'n)}}$ may tend to a finite limit as n tends to infinity for some proper choice of values of a, b, c, a', b'.

Denote $\dfrac{1^1 . 2^2 . 3^3 \ldots n^n}{n^{an^2+bn+c} \, e^{-(a'n^2+b'n)}}$ by u_n.

Then $$\frac{u_n}{u_{n+1}} = n^{\lambda} . \left(1 + \frac{1}{n}\right)^{\mu} . e^{\nu}$$

where $$\lambda = a(n+1)^2 - an^2 + b(n+1) - bn - n - 1$$

$$= n(2a-1) + (a+b-1)$$

$$\mu = a(n+1)^2 + b(n+1) + c - (n+1)$$

$$\nu = -(2a'n + a' + b')$$

Choose a and b such that λ becomes zero for all values of n. This requires that $2a - 1 = 0$, and $a + b - 1 = 0$.

$$\therefore \qquad a = \tfrac{1}{2} \text{ and } b = \tfrac{1}{2}.$$

Then $\mu = \frac{1}{2}(n+1)^2 + \frac{1}{2}(n+1) - (n+1) + c = \dfrac{n(n+1)}{2} + c.$

$$\therefore \qquad \frac{u_n}{u_{n+1}} = \left(1 + \frac{1}{n}\right)^{\frac{n(n+1)}{2} + c} \times e^{-(2a'n+a'+b')}$$

Next choose c, a' and b' such that $\dfrac{u_n}{u_{n+1}}$ is greater than 1 for $n \geqslant m$ (a finite integer) and

$$\underset{n \to \infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = 1$$

$$\log \frac{u_n}{u_{n+1}} = \left\{ \frac{n(n+1)}{2} + c \right\} \log \left(1 + \frac{1}{n}\right) - (2a'n + a' + b'),$$

Since $\dfrac{u_n}{u_{n+1}} \to 1$ as $n \to \infty$ $\underset{n\to\infty}{\text{Lt}} \log \dfrac{u_n}{u_{n+1}} = 0.$

But $\log \dfrac{u_n}{u_{n+1}} = \left\{\dfrac{n(n+1)}{2} + c\right\}\left\{\dfrac{1}{n} - \dfrac{1}{2n^2} + \dfrac{1}{3n^3} - \dots\right\}$

$$-(2a'n + a' + b'),$$

i.e., $= \left\{\dfrac{n+1}{2} + \dfrac{c}{n} - \dfrac{n+1}{4n} - \dfrac{c}{2n^2} + \dfrac{n+1}{6n^2} + \dfrac{c}{3n^3} \dots \dots\right\}$

$$-(2a'n + a' + b').$$

Choose a', b' and c such that the co-efficients of n and $1/n$ and also the constant term becomes zero;

i.e., $\tfrac{1}{2} - 2a' = 0$; $\tfrac{1}{2} - \tfrac{1}{4} - a' - b' = 0$; $c - \tfrac{1}{4} + \tfrac{1}{6} \equiv 0.$

$\therefore$ $a' = \tfrac{1}{4}$, $b' = 0$ and $c = \tfrac{1}{12}$.

Therefore the suitable values for a, b, c and a', b' are found to be equal to $\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{12}$ and $\tfrac{1}{4}, 0$ respectively.

Now $\log\left(\dfrac{u_n}{u_{n+1}}\right) = O\left(\dfrac{1}{n^2}\right)$. $\therefore$ $\underset{n\to\infty}{\text{Lt}} \log\left(\dfrac{u_n}{u_{n+1}}\right) = 0.$

Hence $\dfrac{u_n}{u_{n+1}} \to 1$ as $n \to \infty$.

Also $\text{og}\,\dfrac{u_n}{u_{n+1}} = \left(\dfrac{n^2}{2} + \dfrac{n}{2} + \dfrac{1}{12}\right)\log\left(1 + \dfrac{1}{n}\right) - \dfrac{n}{2} - \dfrac{1}{4}.$

$\therefore$ $\log \dfrac{u_n}{u_{n+1}} > \left(\dfrac{n^2}{2} + \dfrac{n}{2} + \dfrac{1}{12}\right) \times$

$$\left\{\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \frac{1}{5n^5} - \frac{1}{6n^6}\right\} - \frac{n}{2} - \frac{1}{4},$$

i.e., $> \dfrac{n}{2} + \dfrac{1}{2} + \dfrac{1}{12n} - \dfrac{1}{4} - \dfrac{1}{4n} - \dfrac{1}{24n^2} + \dfrac{1}{6n} + \dfrac{1}{6n^2} + \dfrac{1}{36n^3} - \dfrac{1}{8n^2}$

$$-\frac{1}{8n^3} - \frac{1}{48n^4} + \frac{1}{10n^3} + \frac{1}{10n^4} + \frac{1}{60n^5}$$

$$-\frac{1}{12n^4} - \frac{1}{12n^5} - \frac{1}{72n^6} - \frac{n}{2} - \frac{1}{4}.$$

$$\therefore \quad \log \frac{u_n}{u_{n+1}} > \frac{1}{360n^3} - \frac{1}{240n^4} - \frac{1}{15n^5} - \frac{1}{72n^6},$$

$$i.e., \quad > \quad \frac{1}{n^3}\left\{\frac{1}{360} - \frac{1}{240n} - \frac{1}{15n^2} - \frac{1}{72n^3}\right\},$$

$$i.e., \quad > \quad \frac{1}{720n^6}\{2n^3 - 3n^2 - 48n - 10\}.$$

The expression $2n^3 - 3n^2 - 48n - 10$ remains positive for all values of n greater than or equal to 6 and also increases with n for $n \geqslant 4$.

$$\therefore \quad \log \frac{u_n}{u_{n+1}} > \frac{1}{720n^6}(2n^3 - 3n^2 - 48n - 10) > 0 \text{ for } n \geqslant 6.$$

$$\therefore \quad \frac{u_n}{u_{n+1}} > 1 \text{ when } n \geqslant 6,$$

or

$$u_6 > u_7 > u_8 \ldots > u_n > u_{n+1} > \ldots$$

$\therefore$ $\{u_n\}$ is a monotonic decreasing sequence. Therefore it must have a limit finite or $-\infty$; but it cannot be $-\infty$, since $u_n > 0$ foı all n, *i.e.*, $\{u_n\}$ is bounded below.

Next it can be shown that the limit is greater than zero.

$$\log \frac{u_n}{u_{n+1}} < \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}\right)\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3}\right) - \frac{n}{2} - \frac{1}{4},$$

$$\therefore \quad \log \frac{u_n}{u_{n+1}} < \frac{n}{2} + \frac{1}{2} + \frac{1}{12n} - \frac{1}{4} - \frac{1}{4n} - \frac{1}{24n^2} + \frac{1}{6n} + \frac{1}{6n^2} + \frac{1}{36n^3},$$

$$i.e., \quad < \frac{1}{8n^2} + \frac{1}{36n^3} \qquad \text{for all values of } n.$$

$$\therefore \quad \left(\log \frac{u_1}{u_2} + \log \frac{u_2}{u_3} + \ldots + \log \frac{u_n}{u_{n+1}}\right) < \frac{1}{8}\sum_{1}^{n}\frac{1}{k^2} + \frac{1}{36}\sum_{1}^{n}\frac{1}{k^3},$$

$$i.e., \qquad \log \frac{u_1}{u_{n+1}} < \frac{1}{8}\sum_1^n \frac{1}{k^2} + \frac{1}{36}\sum_1^n \frac{1}{k^3}.$$

$$\therefore\ \log \frac{u_1}{\alpha} \leqslant \frac{1}{8}\left(\frac{1}{1^2}+\frac{1}{2^2}+\ldots+\ldots \text{ to inf.}\right)+\frac{1}{36}\left(\frac{1}{1^3}+\frac{1}{2^3}+\ldots+\ldots\right),$$

$$i.e., \qquad < \frac{1}{8}\cdot\frac{2}{1}+\frac{1}{36}\cdot\frac{3}{2}=\frac{1}{4}+\frac{1}{24},$$

where α stands for the limit of u_n as n tends to infinity.

$$\therefore \qquad \frac{u_1}{\alpha} < e^{\frac{1}{4}+\frac{1}{24}} \text{ or } \frac{\alpha}{u_1} > e^{-\frac{1}{4}-\frac{1}{24}},$$

$$\text{or} \qquad \alpha > u_1 e^{-\frac{1}{4}-\frac{1}{24}} = e^{-\frac{1}{24}}, \quad \text{since } u_1 = e^{\frac{1}{4}}.$$

This shows that $u_n \to \alpha > 0$ as $n \to \infty$.

Thus we have proved that u_n has a finite limit which is not zero and is greater than $e^{-\frac{1}{24}}$ and less than u_6,

$$i.e., \qquad e^{-\frac{1}{24}} < \alpha < u_6.$$

To find the approximate value of α we may proceed as follows.

It is proved that $u_n > u_{n+1}$ when $n \geqslant 6$. But it may be verified that the inequality is true for all positive integral values of n, by actually finding out the values of u_1, u_2, u_3, u_4, u_5 and u_6. We know that $u_1 = e^{1/4} > \alpha$,

$$i.e., \qquad u_1 > u_2 > u_3 > u_4 \ldots > u_n > u_{n+1} \ldots > \alpha.$$

$$\log \frac{u_n}{u_{n+1}} = \left(\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}\right) \log\left(1+\frac{1}{n}\right) - \frac{n}{2} - \frac{1}{4}$$

$$< \left(\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}\right)\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\frac{1}{4n^4}+\frac{1}{5n^5}-\frac{1}{6n^6}+\frac{1}{7n^7}\right)$$

$$-\frac{n}{2}-\frac{1}{4}$$

i.e., $$< \frac{1}{360}\cdot\frac{1}{n^3} - \frac{1}{240}\cdot\frac{1}{n^4} + \frac{1}{210}\cdot\frac{1}{n^5} + \frac{29}{504}\cdot\frac{1}{n^6} + \cdot\frac{1}{84}\cdot\frac{1}{n^7}.$$

$$\therefore \log\frac{u_1}{\alpha} < \frac{1}{360}\sum_1^\infty \frac{1}{n^3} - \frac{1}{240}\sum_1^\infty \frac{1}{n^4}$$

$$+ \frac{1}{210}\sum_1^\infty \frac{1}{n^5} + \frac{29}{504}\sum_1^\infty \frac{1}{n^6} + \frac{1}{84}\sum_1^\infty \frac{1}{n^7}.$$

Therefore, $\alpha > u_1 e^\lambda$, where

$$\lambda = -\frac{1}{360}\zeta(3) + \frac{1}{240}\zeta(4) - \frac{1}{210}\zeta(5) - \frac{29}{504}\zeta(6) - \frac{1}{84}\zeta(7).$$

and $$\zeta(p) = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots + \frac{1}{n^p} + \dots \text{ to inf.}$$

$\therefore$ $\alpha < e^{\lambda + \frac{1}{4}}$ because $u_1 = e^{\frac{1}{4}}$.

Taking the approximate values of $\zeta(3)$, $\zeta(4)$, etc., and simplifying it can be shown that α is greater than 1.194 (approximately).

u_1 is nearly equal to 1.284 and u_{20} is approximately equal to 1.278.

Also $\alpha < u_r$ for any positive integral value of r.

$$\therefore \quad 1.278 > \alpha > 1.194.$$

Better limits for α may be found.

A deduction from the above result.

It is known that

$$\mathop{\mathrm{Lt}}_{n\to\infty} \frac{u_{n+1}}{u_n} = \mathop{\mathrm{Lt}}_{n\to\infty} (u_n)^{1/n}$$

If $u_n = \dfrac{1^1\, 2^2 . 3^3 \dots n^n}{n^{\frac{1}{2}n^2+\frac{1}{2}n+\frac{1}{12}}\ e^{-\frac{1}{4}n^2}}$, then $\mathop{\mathrm{Lt}}\limits_{n\to\infty} u_n = \alpha$,

and $\underset{n\to\infty}{\text{Lt}} \dfrac{u_{n+1}}{u_n} = 1. \quad \therefore \quad \underset{n\to\infty}{\text{Lt}} (u_n)^{1/n}$ is also equal to 1.

But $$u_n = \frac{(n\,!)^n}{1\,!\,2\,!\,3\,!\,\ldots\,(n-1)\,!} \cdot \frac{e^{\frac{1}{4}n^2}}{n^{\frac{1}{2}n^2+\frac{1}{2}n+\frac{1}{12}}}.$$

$$\therefore \quad (u_n)^{1/n} = \frac{1}{\{1!\,2!\,3!\,(n-1)!\}^{1/n}} \cdot \frac{n\,!\;e^{n/4}}{n^{\frac{1}{2}n+\frac{1}{2}+1/12n}}$$

$$= \frac{1}{\{1!\,2!\,3!\,\ldots\,(n-1)!\}^{1/n}} \cdot \frac{n\,!\;n^{\frac{n}{2}}\;n^{-\frac{1}{12n}}}{n^{n+\frac{1}{2}}\,e^{-n}\,.\,e^{\frac{3}{4}n}}$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} (u_n)^{n/1} =$$

$$\underset{n\to\infty}{\text{Lt}} \frac{n^{n/2}\,.\,e^{-3n/4}}{\{1!\,2!\,3!\,\ldots\,(n-1)!\}^{1/n}}\;\underset{n\to\infty}{\text{Lt}} \frac{n\,!}{n^{n+\frac{1}{2}}\,e^{-n}}\,.\;\underset{n\to\infty}{\text{Lt}}\; n^{-1/12n}.$$

Now, $\underset{n\to\infty}{\text{Lt}}\; n^{-1/12n} = \underset{n\to\infty}{\text{Lt}}\; e^{-\log n/12n} = e^0 = 1,$

and $$\underset{n\to\infty}{\text{Lt}} \frac{n\,!}{n^{n+\frac{1}{2}}\,e^{-n}} = \sqrt{2\pi} \qquad \text{(Stirling's Theorem).}$$

Also $$\underset{n\to\infty}{\text{Lt}}\; u_n{}^{1/n} = \underset{u\to\infty}{\text{Lt}} \frac{u_{n+1}}{u_n} = 1.$$

$$\therefore \quad 1 = \underset{n\to\infty}{\text{Lt}}\,.\,\frac{n^{n/2}\,e^{-3n/4}}{\{1!\,2!\,\ldots\,(n-1)!\}^{1/n}} \times \sqrt{2\pi},$$

or $$\underset{n\to\infty}{\text{Lt}}\,.\,\frac{\{1!\,2!\,3!\,\ldots\,(n-1)!\}^{1/n}}{n^{n/2}\,e^{-3n/4}} = \sqrt{2\pi}.$$

The same result may be written in the form

$$\text{Lt}\,\frac{\{\Gamma(2)\,.\,\Gamma(3)\,.\,\Gamma(4)\,\ldots\,\Gamma(n)\}^{1/n}}{n^{n/2}\,e^{-3n/4}} = \sqrt{2\pi}.$$

Therefore when n is large the value of

$$\{\Gamma(2).\Gamma(3).\Gamma(4)\ldots\Gamma(n)\}^{1/n} = \sqrt{2\pi}\,.\,n^{n/2}\,.\,e^{-3n/4}.$$

A similar method may successfully be employed to find an approximate value of

$$x^x.(x+1)^{x+1}.(x+2)^{x+2}\ldots(x+n)^{x+n}$$

for large values of n; or in general an attempt may be made to find the order of the expression

$$x^{x^p}.(x+a)^{(x+a)^p}\ldots(x+an)^{(x+an)^p}$$

for different values of x, a and p. For the latter, from the consideration of the curve $y = x^p \log x$, the limits between which the expression

$$x^{x^p}.(x+a)^{(x+a)^p}\ldots(x+an)^{(x+an)^p}$$

lies may be obtained.

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M. N. NARASIMHA IYENGAR.

Solutions.

Question 1520.

(M. Bhimasena Rao):—Show that the determinants of the third order contained in the matrix

$$\begin{vmatrix} \dfrac{\operatorname{sn}(2u-\theta)}{\operatorname{sn}\theta} & \dfrac{\operatorname{cn}(2u-\theta)}{\operatorname{cn}\theta} & \dfrac{\operatorname{dn}(2u-\theta)}{\operatorname{dn}\theta} & 1 \\ \dfrac{\operatorname{sn}(2\theta-u)}{\sin u} & \dfrac{\operatorname{cn}(2\theta-u)}{\operatorname{cn} u} & \dfrac{\operatorname{dn}(2\theta-u)}{\operatorname{dn} u} & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

all vanish.

Solution by K. V. Vedantam.

Let

$$P_1 \equiv x\,\frac{\operatorname{sn}(2u-\theta)}{\operatorname{sn}\theta} + y\,\frac{\operatorname{cn}(2u-\theta)}{\operatorname{cn}\theta} + z\,\frac{\operatorname{dn}(2u-\theta)}{\operatorname{dn}\theta} + 1 = 0;$$

$$P_2 \equiv x\,\frac{\operatorname{sn}(2\theta-u)}{\operatorname{sn} u} + y\,\frac{\operatorname{cn}(2\theta-u)}{\operatorname{cn} u} + z\,\frac{\operatorname{dn}(2\theta-u)}{\operatorname{dn} u} + 1 = 0;$$

and $P_3 \equiv x + y + z + 1 = 0;$

be three planes.

Then the plane through the origin and the line of intersection of P_1 and P_3 is

$$x\,\frac{\operatorname{sn}(2u-\theta)-\operatorname{sn}\theta}{\operatorname{sn}\theta} + y\,\frac{\operatorname{cn}(2u-\theta)-\operatorname{cn}\theta}{\operatorname{sn}\theta} + z\,\frac{\operatorname{dn}(2u-\theta)-\operatorname{dn}\theta}{\operatorname{dn}\theta} = 0$$

or

$$2x\,\frac{\operatorname{sn}(u-\theta)\operatorname{cn} u\operatorname{dn} u}{\operatorname{sn}\theta} - 2y\,\frac{\operatorname{sn}(u-\theta)\operatorname{dn}(u-\theta)\operatorname{sn} u\operatorname{dn} u}{\operatorname{cn}\theta} - 2k^2 z\,\frac{\operatorname{sn}(u-\theta)\operatorname{cn}(u-\theta)\operatorname{sn} u\operatorname{cn} u}{\operatorname{dn}\theta} = 0.$$

i.e., on dividing by $2\operatorname{sn}(u-\theta)\operatorname{sn} u\operatorname{cn} u\operatorname{dn} u$,

$$P_4 \equiv \frac{x}{\operatorname{sn}\theta\operatorname{sn} u} - \frac{y\operatorname{dn}(u-\theta)}{\operatorname{cn}\theta\operatorname{cn} u} - \frac{k^2 z\operatorname{cn}(u-\theta)}{\operatorname{cn}\theta\operatorname{cn} u} = 0.$$

As P_4 is symmetrical in θ and u, it also passes through the line of intersection of P_2 and P_3. So, P_1, P_2, P_3 have a common line. But this is just the condition for the vanishing of the determinants of the third order contained in the matrix.

Question 1524.

(V. RAMASWAMI AIYAR) :—Let $f(x)$ be any function of x; and let $f(a, b, c, \ldots k)$ mean $\Sigma f(a)/(a-b)(a-c)\ldots(a-k)$.

If
$$f_1(x) \equiv f(a_1, a_2, \ldots a_r, x)$$

show that
$$f_1(b_1, b_2, \ldots b_s) = f(a_1, a_2 \ldots a_r, b_1 \ldots b_s).$$

Solutions by N. Sundaram Aiyar.

$f(a_1, a_2, \ldots a_r, b_1, \ldots b_s)$ is obviously the co-efficient of $1/x$ in the expansion, in descending powers of x,

of
$$\frac{f(x)}{(x-a_1)(x-a_2)\ldots(x-a_r)(x-b_1)(x-b_2)\ldots(x-b_s)},$$

i.e., of
$$\frac{f(x)}{(x-a_1)\ldots(x-a_r)}\cdot\Sigma\frac{1}{(x-b_1)(b_1-b_2)\ldots(b_1-b_s)}.$$

But the co-efficient of $1/x$ in

$$\frac{f(x)}{(x-a_1)(x-a_2)\ldots(x-a_u)(x-b_1)}$$

is clearly by the notation of the question $f(a_1, a_2, \ldots a_r, b_1)$!.

Hence

$$f(a_1, a_2, \ldots a_r, b_1, \ldots b_s) = \Sigma\frac{f(a_1, a_2, \ldots a_r, b_k)}{(b_k-b_1)(b_k-b_2)\ldots(b_k-b_s)}$$

where in the denominator the zero factor is excluded. This may be written

$$\Sigma\frac{f_1(b_k)}{(b_k-b_1)(b_k-b_2)\ldots(b_k-b_3)} = f_1(b_1, b_2, \ldots b_s).$$

Also solved by S. Ramaswami Aiyar.

Question 1526.

(HUKAM CHAND):—The equations of three conics being given in the forms

$$a_rx^2 + b_ry^2 + c_rz^2 + d_rw^2 = 0 \qquad (r = 1, 2, 3)$$

where $x + y + z + w = 0$, show that the straight line

$$lx + my + nz + pw = 0$$

will be cut in involution by them if

$$\Sigma \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix} (l - m)(l - n)(l - p) = 0.$$

(Clifford: *Col. Papers*, p. 624).

Solution by* (1) *the Proposer and* (2) *by B. Ramamurti.

(1) From the equations

$$x + y + z + w = 0, \qquad \ldots \text{(i)}$$

$$lx + my + nz + pw = 0 \qquad \ldots \text{(ii)}$$

we get

$$\lambda_1 y = -(\lambda_2 z + \lambda_3 w) \qquad \ldots \text{(iii)}$$

$$\lambda_1 x = (\lambda_4 z + \lambda_5 w) \qquad \ldots \text{(iv)}$$

where $l - m = \lambda_1,\ l - n = \lambda_2,\ l - p = \lambda_3$

$m - n = \lambda_4,\ m - p = \lambda_5,\ n - p = \lambda_6.$

Eliminating x and y between the equations (iii) and (iv) and the equations of the conics, we get the lines

$$z^2(a_r\lambda_4^2 + b_r\lambda_2^2 + c_r\lambda_1^2) + 2zw(a_r\lambda_4\lambda_5 + b_r\lambda_2\lambda_3)$$
$$+ w^2(a_r\lambda_5^2 + b_r\lambda_3^2 + d_r\lambda_1^2) = 0 \qquad (r = 1, 2, 3)$$

which join the intersections of the conics and the line

$$lx + my + nz + pw = 0$$

with the vertex A of the quadrilateral of reference. These lines will be in involution, if

$$\begin{vmatrix} a_1\lambda_4^2+b_1\lambda_2^2+c_1\lambda_1^2 & a_1\lambda_4\lambda_5+b_1\lambda_2\lambda_3 & a_1\lambda_5^2+b_1\lambda_3^2+d_1\lambda_1^2 \\ a_2\lambda_4^2+b_2\lambda_2^2+c_2\lambda_1^2 & a_2\lambda_4\lambda_5+b_2\lambda_2\lambda_3 & a_2\lambda_5^2+b_2\lambda_3^2+d_2\lambda_1^2 \\ a_3\lambda_4^2+b_3\lambda_2^2+c_3\lambda_1^2 & a_3\lambda_4\lambda_5+b_3\lambda_2\lambda_3 & a_3\lambda_5^2+b_3\lambda_3^2+d_3\lambda_1^2 \end{vmatrix}$$

vanishes.

Border the determinant by two rows

$$\begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{matrix}$$

and two columns

$$\begin{matrix} 1 & 0 \\ 0 & 1 \\ a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3. \end{matrix}$$

In the bordered determinant subtract from the third column the sum of λ_4^2 times the first and λ_2^2 times the second column; from the fourth, the sum of $\lambda_4\lambda_5$ times the first and $\lambda_2\lambda_3$ times the second, and from the fifth the sum of λ_5^2 of the first aud λ_3^2 times the second. Then divide the third and fifth columns by λ_1^2 and multiply the first and second rows by $-\lambda_1^2$ each: then

$$\Delta=\lambda_1^2\begin{vmatrix} -\lambda_1^2 & 0 & \lambda_4^2 & \lambda_4\lambda_5\lambda_1^2 & \lambda_5^2 \\ 0 & -\lambda_1^2 & \lambda_2^2 & \lambda_2\lambda_3\lambda_1^2 & \lambda_5^2 \\ a_1 & b_1 & c_1 & 0 & d_1 \\ a_2 & b_2 & c_2 & 0 & d_2 \\ a_3 & b_3 & c_3 & 0 & d_3 \end{vmatrix}$$

Subtract $\lambda_2\lambda_3$ times the first row from $\lambda_4\lambda_5$ times the second and note that $\lambda_3\lambda_4-\lambda_2\lambda_5=-\lambda_1\lambda_6$. We find after expanding in terms of the elements in the fourth column, that the condition becomes

$$\lambda_1^3\begin{vmatrix} \lambda_1\lambda_2\lambda_3 & -\lambda_1\lambda_4\lambda_\iota & \lambda_2\lambda_4\lambda_6 & -\lambda_3\lambda_5\lambda_6 \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix}=0.$$

(2) The three conics and the straight line can be considered as sections of three quadratics and a plane respectively by the plane $x+y+z+w=0$. If (x_1, y_1, z_1, w), (x_2, y_2, z_2, w_2) be two polnts on the line, any point on it is given by

$$\lambda x_2 + x_1, \ \lambda y_2 + y_1, \ \lambda z_2 + z_1, \ \lambda w_2 + w_1$$

where λ is a parameter. If the line is cut in involution by the quadrics, the three quadratics giving the parameters of the three pairs of points of intersection should be linearly connected. The condition for this is

$$\begin{vmatrix} \Sigma a_1 x_1^2 & \Sigma a_1 x_2^2 & \Sigma a_1 x_1 x_2 \\ \Sigma a_2 x_1^2 & \Sigma a_2 x_2^2 & \Sigma a_2 x_1 x_2 \\ \Sigma a_3 x_1^2 & \Sigma a_3 x_{22} & \Sigma a_3 x_1 x^2 \end{vmatrix} = 0$$

where $\Sigma a_1 x_1^2 = a_1 x_1^2 + b_1 y_1^2 + c_1 z^2 + d_1 w_1^2$ and so on.

Expanding the determinants into $4 \times 4 \times 4$ determinants, a typical non-vanishing determinant is

$$\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix} y_1^2 z_2^2 w_1 w_2.$$

The co-efficient of this determinant in the complete expansion is

$$\begin{aligned} & y_1^2 z_2^2 w_1 w_2 - y_1^2 w_2^2 z_1 z_2 \\ & + z_1^2 w_2^2 y_1 y_2 - z_1^2 y_2^2 w_1 w_2 \\ & + w_1^2 y_2^2 z_1 z_2 - w_1^2 z_2^2 - w_1^2 z_2^2 y_1 y_2 \\ = & (y_1 z_2 - y_2 z_1)(z_1 w_2 - z_2 w_1)(w_1 y_2 - w_2 y_1) \qquad \text{... (A)} \end{aligned}$$

Making use of the relations

$$lx_1 + my_1 + nz_1 + pw_1 = 0; \qquad x_1 + y_1 + z_1 + w_1 = 0;$$
$$lx_2 + my_2 + nz_2 + pw_2 = 0; \qquad x_2 + y_2 + z_2 + w_2 = 0;$$

(A) is found to be proportional to $(l - m)(l - n)(l - p)$.

Hence the required condition.

Question 1534.

(K. SATYANARAYANA):—Prove that in all triangles in the non-Euclidean plane in which the base angles are supplementary, the median bisecting the base has the same constant length.

Solution by S. P. Ranganadachar.

Let ABC be the triangle, $\angle B + \angle C = \pi$, and D the mid. point of BC.

Then,

$$\cos\left(\frac{AD}{k}\right) = \cos\left(\frac{a}{2k}\right)\cos\left(\frac{c}{k}\right) + \sin\left(\frac{a}{2k}\right)\sin\left(\frac{c}{k}\right)\cos B \ldots \quad (1)$$

where $\frac{1}{k^2}$ stands as usual for the measure of curvature of space ;

and

$$\cos\left(\frac{AD}{k}\right) = \cos\left(\frac{a}{2k}\right)\cos\left(\frac{b}{k}\right) - \sin\left(\frac{a}{2k}\right)\sin\left(\frac{b}{k}\right)\cos B \ldots \quad (2)$$

since $\angle B + \angle C = \pi$.

Also $$\frac{\sin(a/k)}{\sin A} = \frac{\sin(b/k)}{\sin B} = \frac{\sin(c/k)}{\sin C}.$$

Since $\sin B = \sin C$ it follows that $\sin\frac{b}{k} = \sin\frac{c}{k}$,

i.e., $$\frac{b}{k} = \frac{c}{k} \text{ or } \frac{b}{k} = \pi - \frac{c}{k}.$$

If $\frac{b}{k} = \frac{c}{k}$, the base angles must be equal. Since they are supplementary, we must have

$$\frac{b}{k} = \pi - \frac{c}{k}. \quad \ldots \quad (3)$$

Adding (1) and (2) and using (3), we get

$$2\cos\left(\frac{AD}{k}\right) = 0.$$

$$\therefore \quad AD = \frac{k\pi}{2}, \text{ a constant.}$$

Question 1561.

(S. D. CHOWLA) :—If p is a prime $\equiv 9 \pmod{16}$, prove that

$$\sum_{1}^{\frac{1}{2}(p-1)} \left(\frac{n}{p}\right) \left(\frac{n^8+1}{p}\right) = 0$$

where $\left(\frac{n}{p}\right)$ is the Legendre quadratic residue symbol.

Solution by the Proposer.

$$\sum_{1}^{p-1} \left(\frac{n^9+n}{p}\right) \equiv \sum_{1}^{p-1} (n^9+n)^{\frac{p-1}{2}} \qquad \pmod{p}$$

$$\equiv \sum_{n=1}^{p-1} \sum_{k=0}^{\frac{1}{2}(p-1)} {}_{\frac{1}{2}(p-1)}C_k \, n^{9k} \, n^{\frac{p-1}{2}-k}$$

$$\equiv \sum_{k=0}^{\frac{1}{2}(p-1)} {}_{\frac{1}{2}(p-1)}C_k \left[\sum_{n=1}^{p-1} n^{8k+\frac{1}{2}(p-1)} \right] \qquad \ldots (1)$$

Now

$$\sum_{s=1}^{p-1} s^m \equiv -1 \pmod{p} \text{ if } m \equiv 0 \pmod{\overline{p-1}}$$

$$\sum_{s=1}^{p-1} s^m \equiv 0 \pmod{p} \text{ if } m \not\equiv 0 \pmod{\overline{p-1}}$$

Now

$$8k + \tfrac{1}{2}(p-1) \equiv 0 \pmod{\overline{p-1}}$$

means that $k = \dfrac{(2m+1)(p-1)}{16}.$

This is not possible since $p \equiv 9 \pmod{16}$.

Hence the sum in square brackets in (1) is $\equiv 0 \pmod{p}$, for all k.

Hence

$$\sum_{1}^{p-1} \left(\frac{n^9+n}{p}\right) \equiv 0 \pmod{p}, \text{ and hence } = 0.$$

Jacobsthal (*Diss* Berlin, 1906) has proved

$$\sum_{1}^{p-1} \left(\frac{n^5+1}{p}\right) = 0 \text{ for primes } p \equiv 5 \pmod{8},$$

by another method.

Questions for Solution.

Proposers of Questions are requested to send their own solutions along with their questions.

1611. (P. GANAPATI):—If L is the perimeter and F the area of any closed convex plane curve, show that

$$\frac{\Delta}{6} \leqslant \frac{F}{L} \leqslant \frac{D}{4}$$

where Δ is the minimum width and D the maximum width of the curve.

Show also that the first inequality becomes an equality only when the curve is an equilateral triangle, and the second when it is a circle.

1612. (P. GANAPATI):—Show that for any oval with continuous curvature there are at least three pairs of points such that the tangents at each pair are parallel and the ratio of the curvatures is an extremum.

1613. (V. M. TELANG):—Let OA be a diameter of a circle B an extremity of the perpendicular diameter, and BC a chord equal in length to the radius. Join AC and measure AD = AB. Prove that the angle AOD thus constructed is a very close approximation to $40°$.

1614. (K. J. SANJANA, M.A.):—(1) Two known stars are observed to lie on the horizon of a place at one and the same instant: show how to determine the latitude of the place.

(2) Deduce from the preceding the condition that three points on a sphere should lie on one and the same great circle.

(3) The stars Vega (α Lyrae, 0.14) and Achernar (α Eridani, 0.60) are observed to set at one and the same instant: determine to the nearest minute the latitude of the place of observation. [The latitude will be found to be North; in exactly the same South latitude the stars rise at the same instant.]

1615. (P. JAGANNATHAN):—Prove that

$$\frac{2\,(3n)!}{n!\,(n+1)!\,(n+2)!}$$

is an integer for all positive integral values of n.

Examine whether the factor 2 in the numerator may be omitted without affecting the result.

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1 Acta Mathematica, **58**, 1, 2.
2 American Journal of Mathematics, **54**, 1 (2 copies).
3 American Mathematical Monthly, **37**, 12, **38**, 10, **39**. 1, 2, (3 copies each).
4 Annales de l'Ecole Normale Superieure, Nos. 11, 12, Nov. & Dec. 1931.
5 Annals of Mathematics, **33**, 1 (2 copies).
6 Astrophysical Journal, **74**, 5.
7 Bulletin of the American Mathematical Society, **38**, 1, 2.
8 Bulletin of the Calcutta Mathematical Society, **23**, 4.
9 Bulletin des Sciences Mathematiques, **56**, Jan. & Feb. 1932 (2 copies each).
10 Crelle's Journal, **167**, 29th Dec. 1931.
11 Japanese Journal of Mathematics, **8**, 3.
12 Mathematical Gazette, **16**, 217 (3 copies).
13 Mathematische Annalen, **105**, 5, **106**, 1.
14 Monthly Notices of the Royal Astronomical Society, **92**, 1, 2, 3.
15 Philosophical Magazine, **13**, 82, 83, 84.
16 Philosophical Transactions of the Royal Society, **230**, 387—411, 413—445.
17 Popular Astronomy, **40**, 1, 2.
18 Proceedings of the Cambridge Philosophical Society, **28**, 1.
19 Proceedings of the London Mathematical Society, **33**, 1, 2, 3, 4 (3 copies each).
20 Proceedings of the Physico-Mathematical Society of Japan **13**, 11, **14**, 1.
21 Proceedings of the Royal Society of London, **134**, 825, **135**, 826.
22 Rendiconti del Circolo Matematico di Palermo, 55 Anno (93).
23 Rendiconti del Seminario Mathematico, **9**, 2.
24 Revista Mathematica, Hispano-Americano, **6**, **7**, 8, 9, 10.
25 Tohoku Mathematical Journal, **35**, 1.
26 Transactions of the Royal Society of South Africa, **20**, 2, 3.

Books.

1 School Mathematics with Answers, L. Crosland.

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